Quantum Simulation
of
Geometry
(and Arithmetics)

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Outline

- Motivation
- Quantum Simulation of Background Geometry
- Quantum Simulation of an Extra Dimension
- Quantum Simulation of Topology
- Primes

Quantum Computation

What?

Why?

When?

Where?

Who?

Topological order

Artificial systems

NV Centers

Optomechanics

Sensors

Quantum Engineering

Control of positions

Qubits superconductores

Control of time

Tensor Networks

Ion traps

Cold Gases

Precision

Quantum Simulation

Computation

Teleportation

Non-locality

Quantum Information

WHAT?

Randomness

Bell

Philosophy

Quantum Mechanics
Elementary Particles, Nuclear Physics, Atomic&Molecular Physics,
Condensed Matter, Quantum Field Theory, Astrophysics,
Quantum Optics, Solid State, …

Description of Nature

Fist Quantum Steps of Human Kind

Validation of theory

- Exact Calculations

  Integrable models (from Hamiltonian), CFT (from symmetry),

  ADS/CFT (from conjecture)

- Approximate methods

  Perturbation theory, toy models, non-perturbative techniques,...

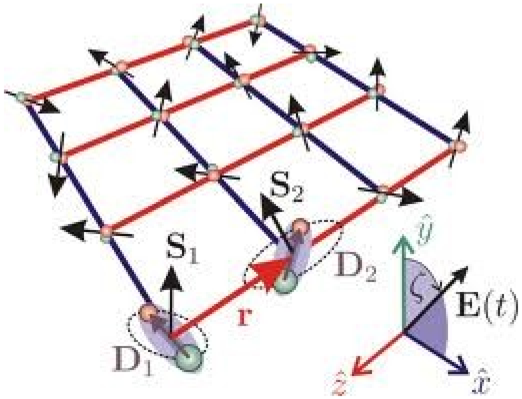
- Numerics

  Monte Carlo, Tensor Networks (MPS, PEPS, MERA,...)

other powerful instruments?

Analogue Simulation

≈



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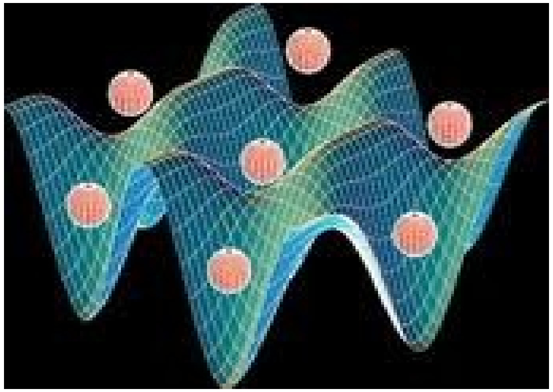


Theory of interest

System under control

$$H\left(\left\{\alpha\right\}\right)\approx H'\left(\left\{\lambda\right\}\right)$$

controlled parameters

Analogies have to be analyzed very critically

Why?

Quantum Simulation is an intelligent window to QC

Quantum Computer

General purpose quantum computation
Shor’s factorization algorithm
Oracle problems, NAND trees, …

Few problems, few algorithms!

Quantum Simulator

Efficient analysis of specific quantum problems
Explore new Physics

Experimentally achievable

Classical Computer

Tensor Networks strategies: PEPS, MERA
Monte Carlo

Not sufficient

Quantum Simulation

Why → What?

- Q Simulation of models beyond classical simulation
- Q Simulation of criticality, frustration, topological order,...
- Q Simulation of non-abelian gauge theories
- Q Simulation of unphysical models, Klein paradox, Zitterbewegung,...
- Q Simulation of gravity, geometry, topology

Where?

- Ion traps
- Cold gases
- Molecules, solids, graphene, …

Quantum Simulation

of

Gravitational Backgrounds

Dirac equation on a square lattice

$$i\left(\gamma^0\partial_0+\gamma^1\partial_1+\gamma^2\partial_2\right)\psi=0$$

Can we simulate the Dirac equation on optical lattices?

Can we simulate curved spaces?

O. Boada, A. Celi, M. Lewenstein, JIL

Dirac Hamiltonian in 2+1 dimensions

$$i\partial_t\psi = H\psi = -i\gamma_0(\gamma_1\partial_1 + \gamma_2\partial_2)\psi$$

$$i\gamma_0\gamma_1 = -i\gamma_2 = \sigma_x \qquad i\gamma_0\gamma_2 = i\gamma_1 = \sigma_y \qquad i\gamma_1\gamma_2 = i\gamma_0 = i\sigma_z$$

$$H\psi = -(\sigma_x\partial_x + \sigma_y\partial_y)\psi = 0$$

$$\mathrm{H} = \int dx\, dy\, \psi^\dagger H \psi$$

Discretized Dirac Hamiltonian

$$\mathrm{H} = \frac{1}{2a}\sum_{m,n}(\psi^\dagger_{m+1,n}(\sigma_x)\psi_{m,n} + \psi^\dagger_{m,n+1}(\sigma_y)\psi_{m,n}) + h.c.$$

SU(2) Fermi Hubbard model

Dirac equation in **curved** space-time

$$D_\mu = \partial_\mu + \frac{1}{2} w_\mu^{ab} \gamma_{ab}$$

$$\gamma^\mu D_\mu \psi = 0$$

$$\gamma^\mu = e_a^\mu \gamma^a$$

$$\gamma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b]$$

If there exists a timelike Killing vector
(time translation invariance in certain coordinates)

➡ there exists *H* conserved and well defined

Sufficient condition $\partial_t g_{\mu\nu} = 0$

$$H = -i \gamma_t \left(\gamma^i \partial_i + \frac{1}{4} \gamma^i w_i^{ab} \gamma_{ab} + \frac{1}{4} \gamma^t w_t^{ab} \gamma_{ab} \right)$$

$$\mathrm{H} = \int \sqrt{-g}\, dx\, dy\, \psi^\dagger \gamma_0 \gamma^t H \psi$$

Rindler space-time

$$ds^2 = -(C\,x)^2 dt^2 + dx^2 + dy^2$$

$$e^0 = |C\,x|\,dt \qquad e^1 = dx \qquad e^2 = dy$$

Steady Rindler observer is an accelerated Minkowski observer

acceleration $\frac{1}{Cx}$ $\xrightarrow[\text{Unruh effect}]{}$ **temperature** $\frac{1}{Cx}$

Rindler is the near horizon limit of Schwarzschild black hole

For any metric of the form

$$ds^2 = -e^{\Phi(x,y)} dt^2 + dx^2 + dy^2$$

The lattice version turns out to be

$$\mathrm{H} = \frac{1}{2a} \sum_{m,n} J_{mn} \left(\psi^{\dagger}_{m+1,n} \sigma_x \psi_{m,n} + \psi^{\dagger}_{m,n+1} \sigma_y \psi_{m,n} \right) + h.c.$$

$$J_{mn} = e^{\Phi(am,an)}$$

geometry = energy cost for jumping to a nearest neighbor

Site dependent couplings!

Discretized Dirac equation in a Rindler space

$$H=\frac{1}{2a}\sum_{m,n} c\,m\left(\psi^{\dagger}_{m+1,n}\sigma_x\psi_{m,n}+\psi^{\dagger}_{m,n+1}\sigma_y\psi_{m,n}\right)+h.c.$$

Experimental options

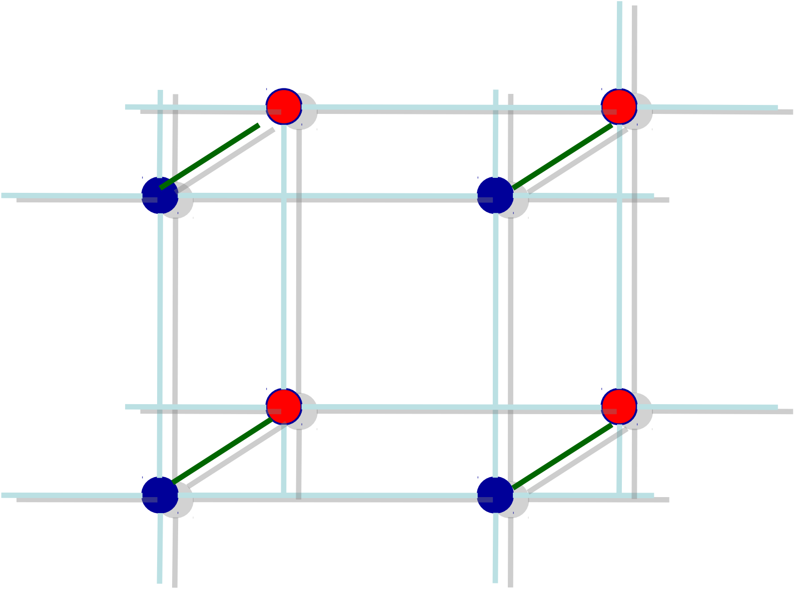
- **superlattice techniques**
- **laser waist**

Quantum Simulation

of

an extra dimension

Connectivity

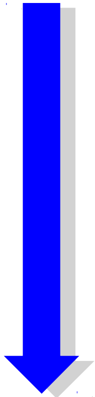


dimensions = connectivity

D+1 dimensions can be simulated in D dimensions
by tuning appropriately the nearest neighbor couplings

O. Boada, A. Celi, Lewenstein, JIL

Dimension



Species

$$H=-J\sum_{\vec{q}}\sum_{j=1}^{D+1} a^{\dagger}_{\vec{q}+\vec{u}_j}\, a_{\vec{q}}+h.c.$$

$$\vec{q}=(\vec{r}\,,\sigma)$$

D + 1

$$H=-J\sum_{r,\sigma}\sum_{j=1}^{D}\left(a^{(\sigma)\dagger}_{\vec{r}+\vec{u}_j}\, a^{(\sigma)}_{\vec{r}}+a^{(\sigma+1)\dagger}_{\vec{r}}\, a^{(\sigma)}_{\vec{r}}\right)+h.c.$$

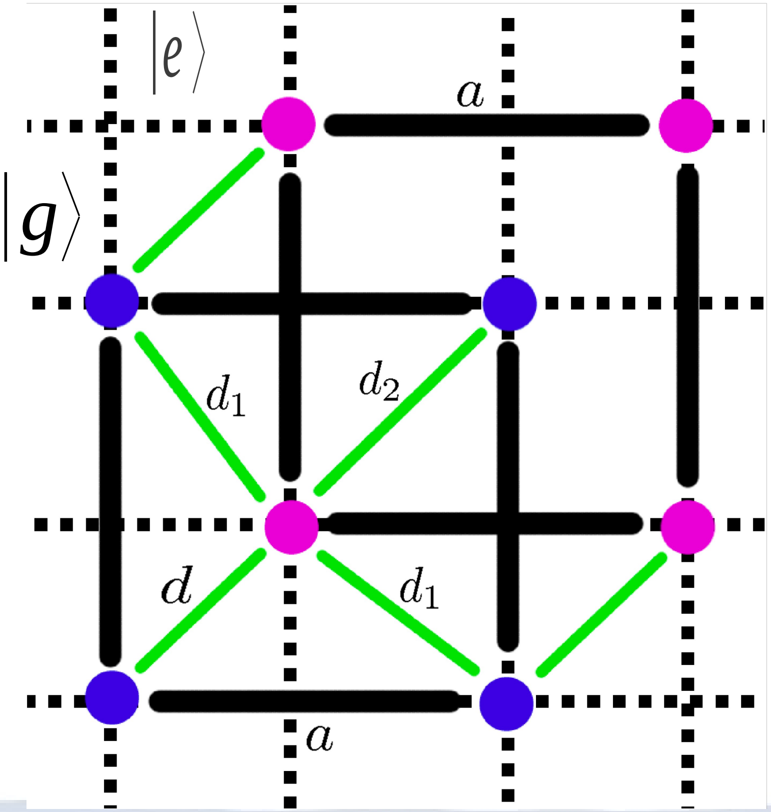
jump

change species

Bivolum

State dependent lattice / On site dressed lattice

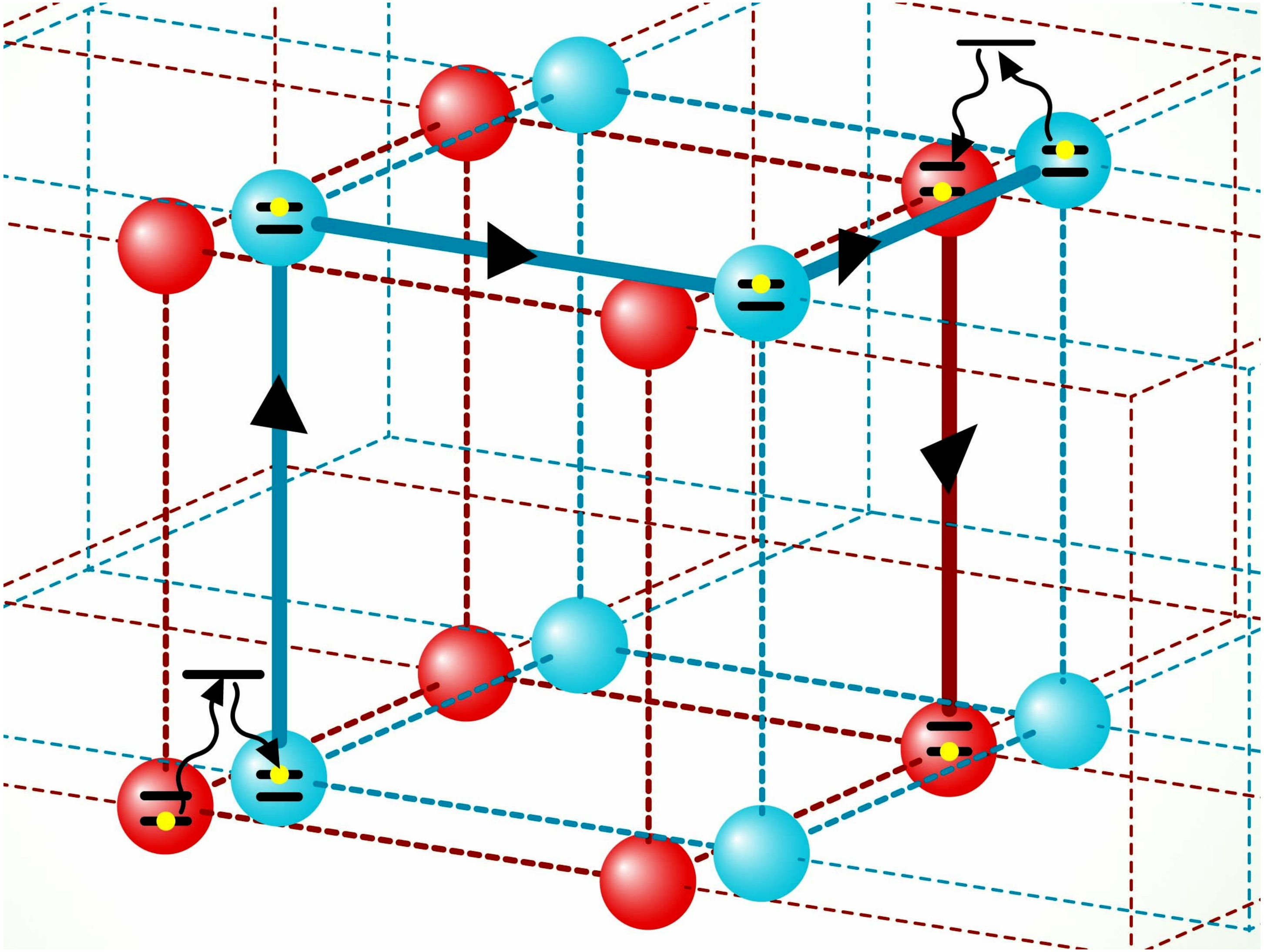
Two 3D sub-lattices are connected via Raman transitions



$$J_{bilayer} = \frac{\Omega}{2} \int d^{2x} \, w^*(\vec{x}) \, w(\vec{x} - \vec{r})$$

Exponential decay of Wannier functions suppresses undesired transitions

Boada, Celi, Lewenstein, JIL PRL



Single particle observable:

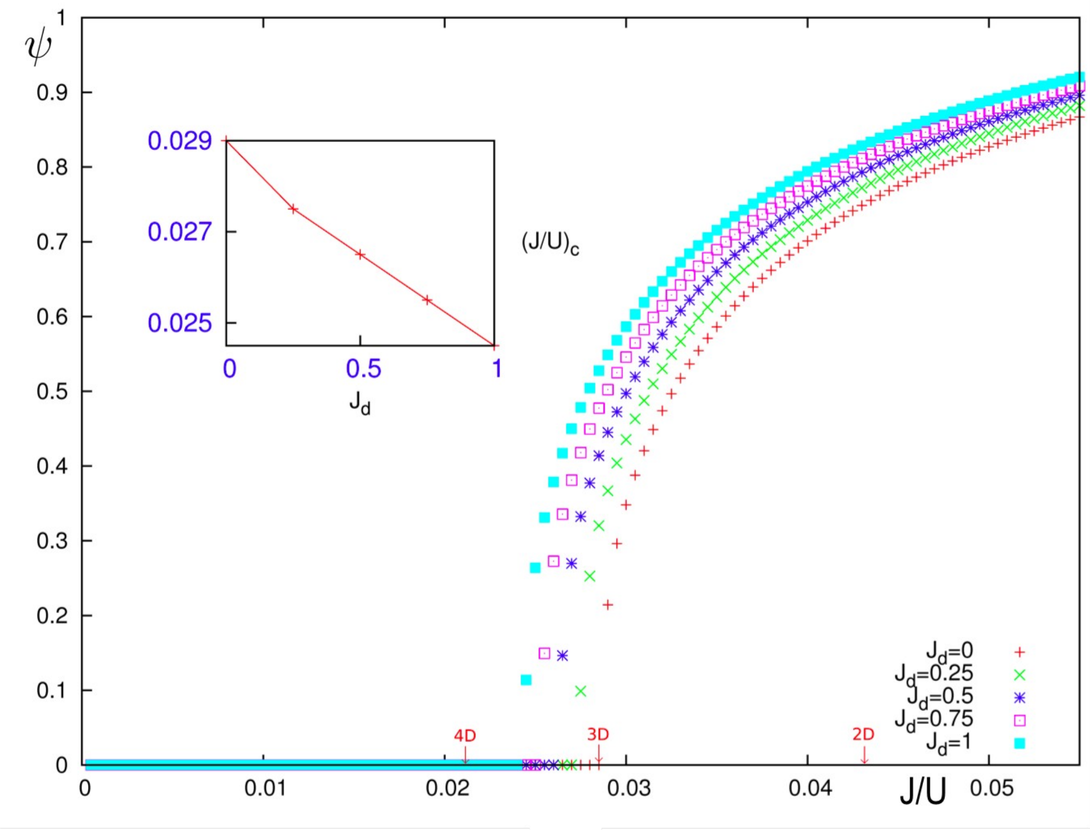
Kaluza-Klein modes

Single particle correlators take contributions from jumps back and forth to other dimensions in the form of exponential (KK) massive corrections

Many-body observables:

Shift of phase transition point

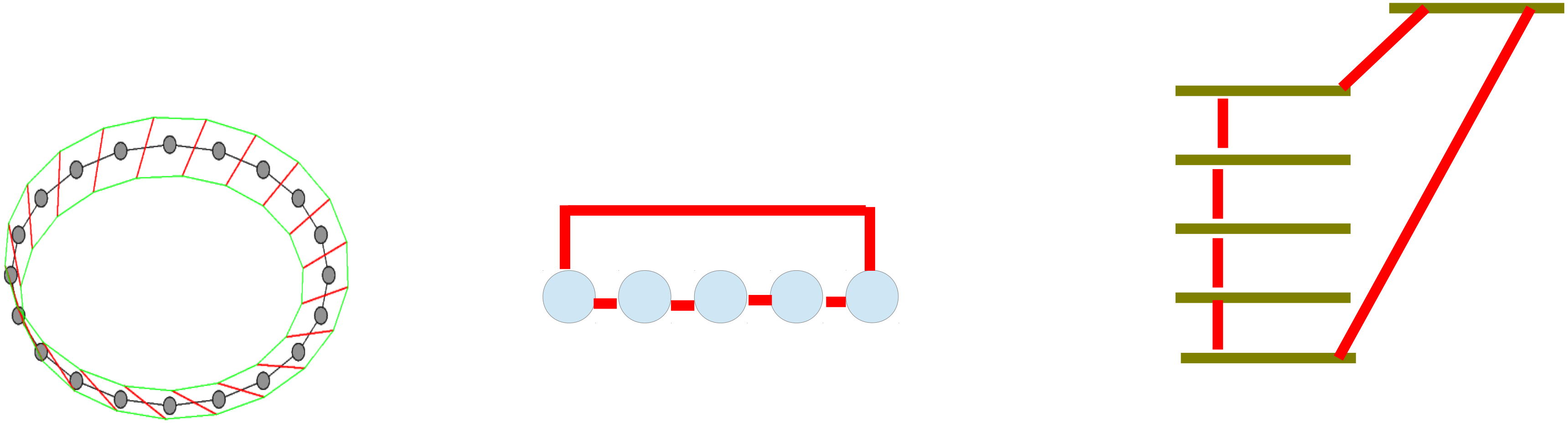
interpolates between dimensions



Quantum Simulation of topology

Quantum Simulation of boundary conditions

Non-local interactions
can be artificially generated



O. Boada, A. Celi, M. Lewenstein, J. Rodríguez-Laguna, JIL

Boundary conditions

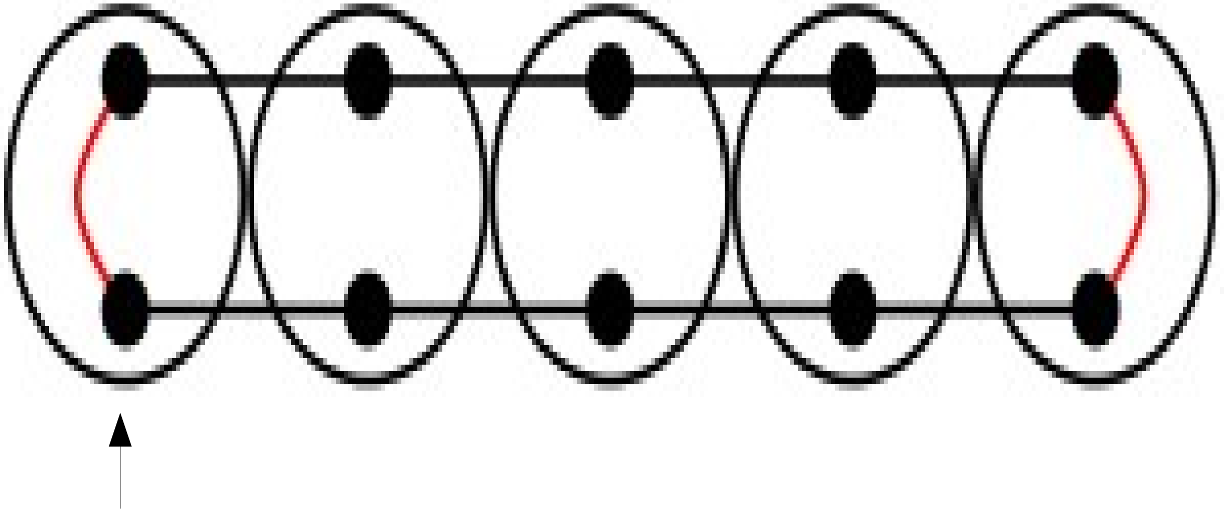
$$H = J \sum_{i=1,\ldots,n-1} \sigma_i^x \sigma_{i+1}^x + J' \sigma_1^x \sigma_n^x$$

Three ways of create the boundary term:

- Bend the physical system on itself
- Create a non-local interaction

- Add an “extra” dimension

Adding extra dimensions (as a species) retains locality of interactions



local interaction

local interaction

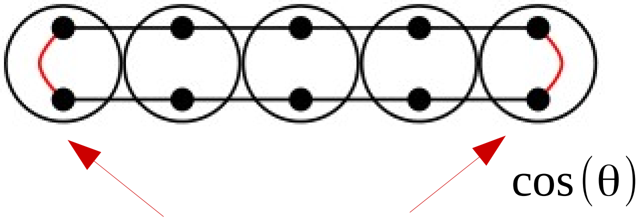
nearest neighbor interaction

species 2

species 1

same site

1-D optical lattice with 2 species can be turned into 1 species on a circle

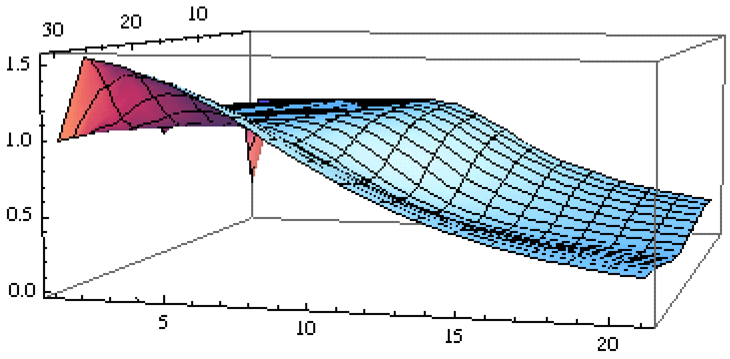


change of species
(ex: local action of Raman lasers at the boundaries)

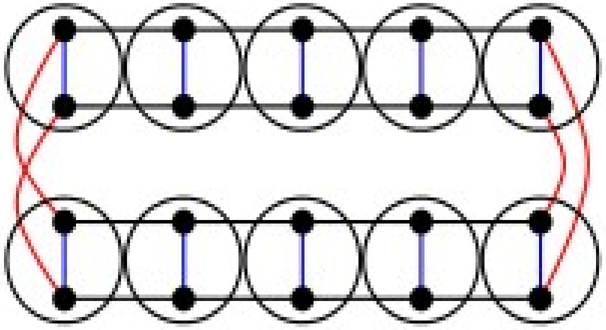
Ex: Frustration from boundary condition on a chain

$$H=\sum_1^n \sigma_i^x \sigma_n^x + \cos(\theta)\, \sigma_n^x \sigma_1^x + \lambda \sum_1^n \sigma_i^z$$

Entaglement entropy jumps

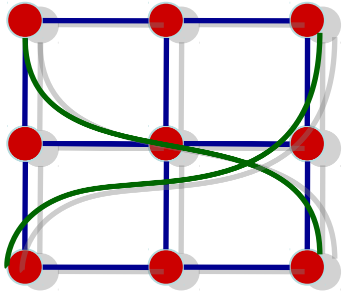


Ladders + Species = Moëbius band



Or 4 species

Torus vs Klein bottle: Hubbard model



On one atom?

Proposals for Quantum Simulation of basic Geometrical concepts

Geometry	Site-dependent coupling
Dimensionality	Connectivity via change of species
Topology	Site depending species coupling

Quantum Simulation is the natural avenue for QM now

What will a full Quantum Computer be used for?

Build a Quantum Computer

↓

Break Classical Cryptography

↓

Use Quantum Cryptography

↓

Idle Quantum Computer?

?

Quantum Counting
for
Arithmetics

?

The Prime State

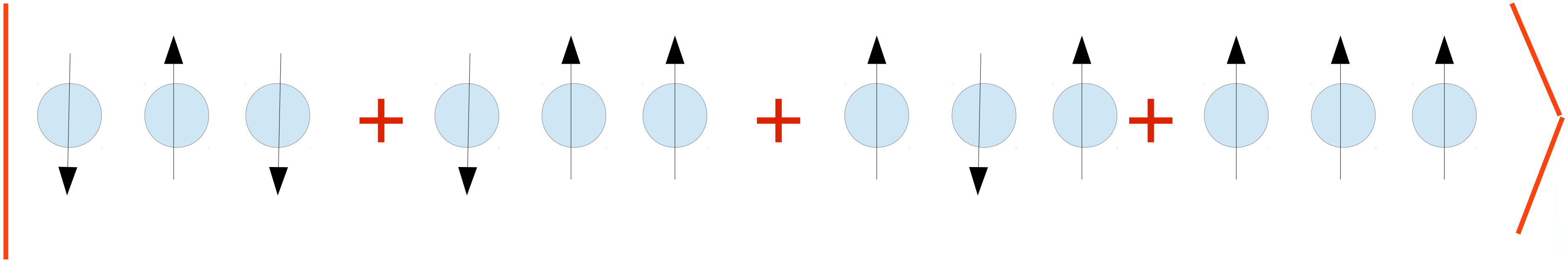
$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p < 2^n \in Primes} |p\rangle$$

$\pi(2^n)$ is the Prime Counting Function

G. Sierra, JIL

Quantum Mechanics allows for the superposition of primes implemented as states of a computational basis

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p<2^n \in Primes} |p\rangle$$



$$|P(3)\rangle = \frac{1}{\sqrt{4}}\left(|2\rangle + |3\rangle + |5\rangle + |7\rangle\right)$$

Gauss, Legendre

Sieve of Eratosthenes

$$\pi(x) \approx \frac{x}{\ln x}$$

Prime Number Theorem

Gauss, Riemann

Hadamard, de la Vallée Poussin

Density of primes 1/log x

$$\pi(x) \approx Li(x)$$

$$Li(x) = \int_2^x \frac{dt}{\log t} \approx \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \ldots$$

$$\pi(10^{24}) = 18\,435\,599\,767\,349\,200\,867\,866$$

Platt (2012)

$$\pi(10^{24}) - \frac{10^{24}}{\ln(10^{24})} = 3.4\,10^{20}$$

$$Li(10^{24}) - \pi(10^{24}) = 1.7\,10^{9}$$

If the Riemann Conjecture is correct, fluctuations are bounded

$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$ If $\zeta(s)=0$ with $0\leq Real(s)\leq 1$ then $Real(s)=\frac{1}{2}$

$$|Li(x)-\pi(x)|<\frac{1}{8\pi}\sqrt{x}\ln x$$

The prime number function will oscillate around the Log Integral infinitely many times
Littlewood, Skewes

A first change of sign is expected for some $x<e^{727.9513468}\ldots$

Could the Prime state be constructed?

Does it encode properties of prime numbers?

What are its entanglement properties?

Could it provide the means to explore Arithmetics?

Entanglement: single qubit reduced density matrices

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{i_{n-1},\dots,i_1,i_0=0,1} p_{i_{n-1}\dots i_1 i_0} |i_{n-1},\dots,i_1,i_0\rangle$$

$$p_{i_{n-1}\dots i_1 i_0} = \begin{cases} 1 & p = i_{n-1}2^{n-1} + \dots + i_0 = prime \\ 0 & otherwise \end{cases}$$

$$\rho^{(1)}_{ab} = \frac{1}{\pi(2^n)} \sum_{i_{n-1},\dots,i_{2,}i_0=0,1} p_{i_{n-1},\dots,i_2,a,i_0} \, p_{i_{n-1},\dots,i_2,b,i_0}$$

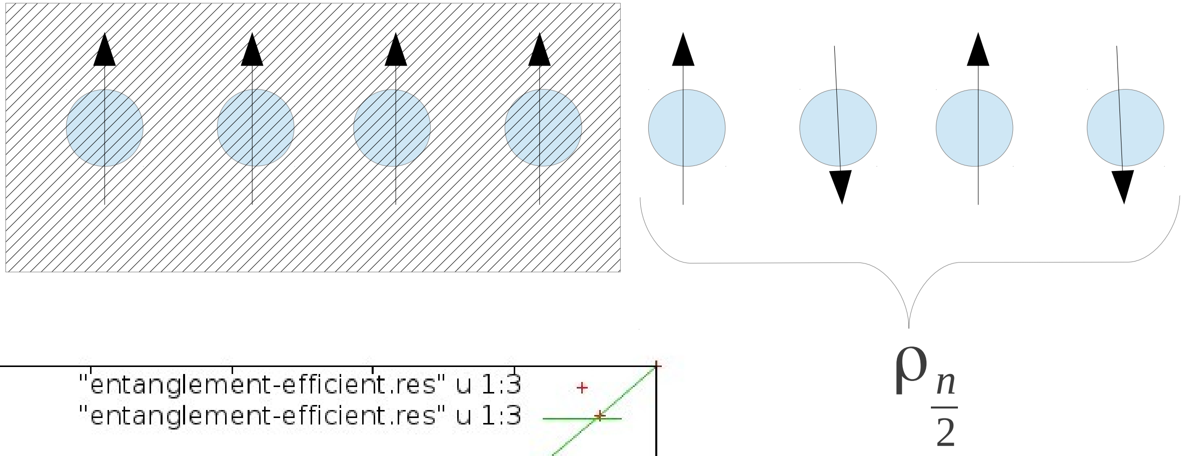
$$\rho^{(1)}_{00} = \frac{\pi_{4,1}(2^n)}{\pi(2^n)} \qquad \rho^{(1)}_{11} = \frac{1+\pi_{4,3}(2^n)}{\pi(2^n)} \qquad \rho^{(1)}_{01} = \frac{\pi^{(1)}_2(2^n)}{\pi(2^n)}$$

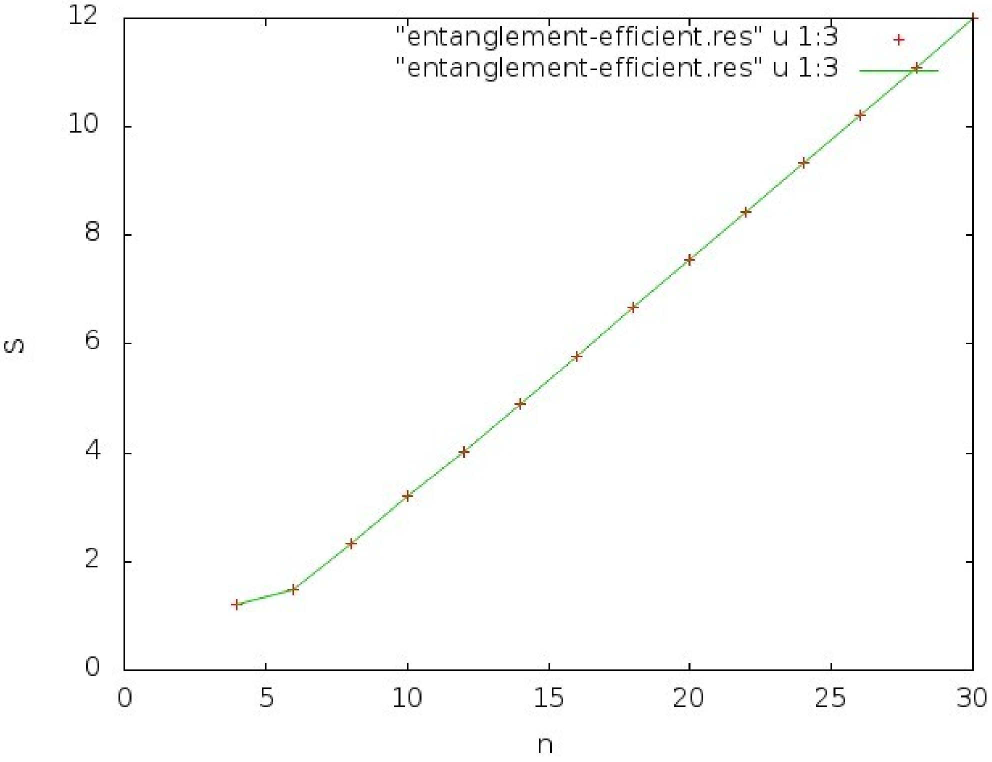
Each element counts sub-series of primes and twin primes

$\pi_{a,b}(x)$ counts the number of primes equal to *a* mod *b*

$\pi^{(1)}_2(x)$ counts twin primes equal to 1 mod 4

Entanglement entropy of the Prime state





“There is entanglement in the Primes”

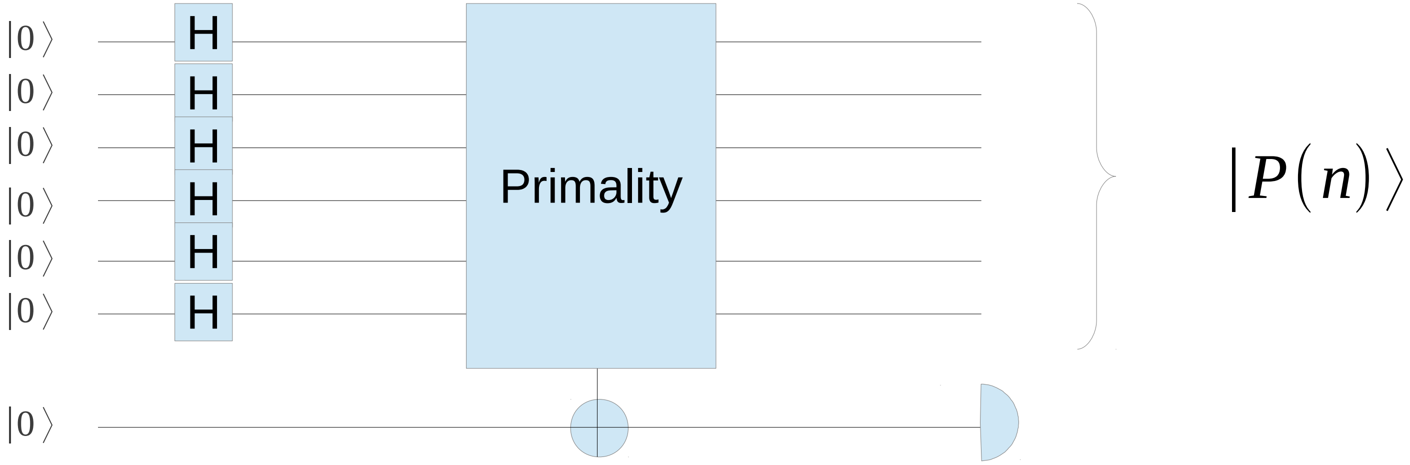
Volume law scaling

$$S \sim .8858\, n + const$$

Scaling of entanglement entropy

$S \sim n - const$	Random states
$S \sim .8858\, n + const$	Prime state
$S \sim n^{\frac{d-1}{d}} + const$	Area law in d-dimensions
$S \sim \frac{c}{3} \log n + const$	Critical scaling in d=1 at quantum phase transitions
$S \sim \log(\xi) = const$	Finitely correlated states away from criticality

Construction of the Prime state



$$U_{primality}\sum_x |x\rangle|0\rangle=|P(n)\rangle|0\rangle+\sum_{c\in composite}|c\rangle|1\rangle$$

$$Prob(|P(n)\rangle)=Prob(ancilla=0)=\frac{\pi(2^n)}{2^n}\approx\frac{1}{n\ln 2}$$

Efficient construction!

Grover construction of the Prime state

$$|\psi_0\rangle = \sum_{x<2^n} |x\rangle = \frac{1}{\pi\left(2^n\right)} \left(\underbrace{\overbrace{\sum_{p\in primes} |p\rangle}^{M} + \sum_{c\in composites} |c\rangle}_{N} \right)$$

calls to Grover

$$R(n) \leq \frac{\pi}{4}\sqrt{\frac{N}{M}} \leq \frac{\pi}{4}\sqrt{n\ln 2}$$

$$\Big\downarrow$$

$$|\psi_f\rangle = |P(n)\rangle$$

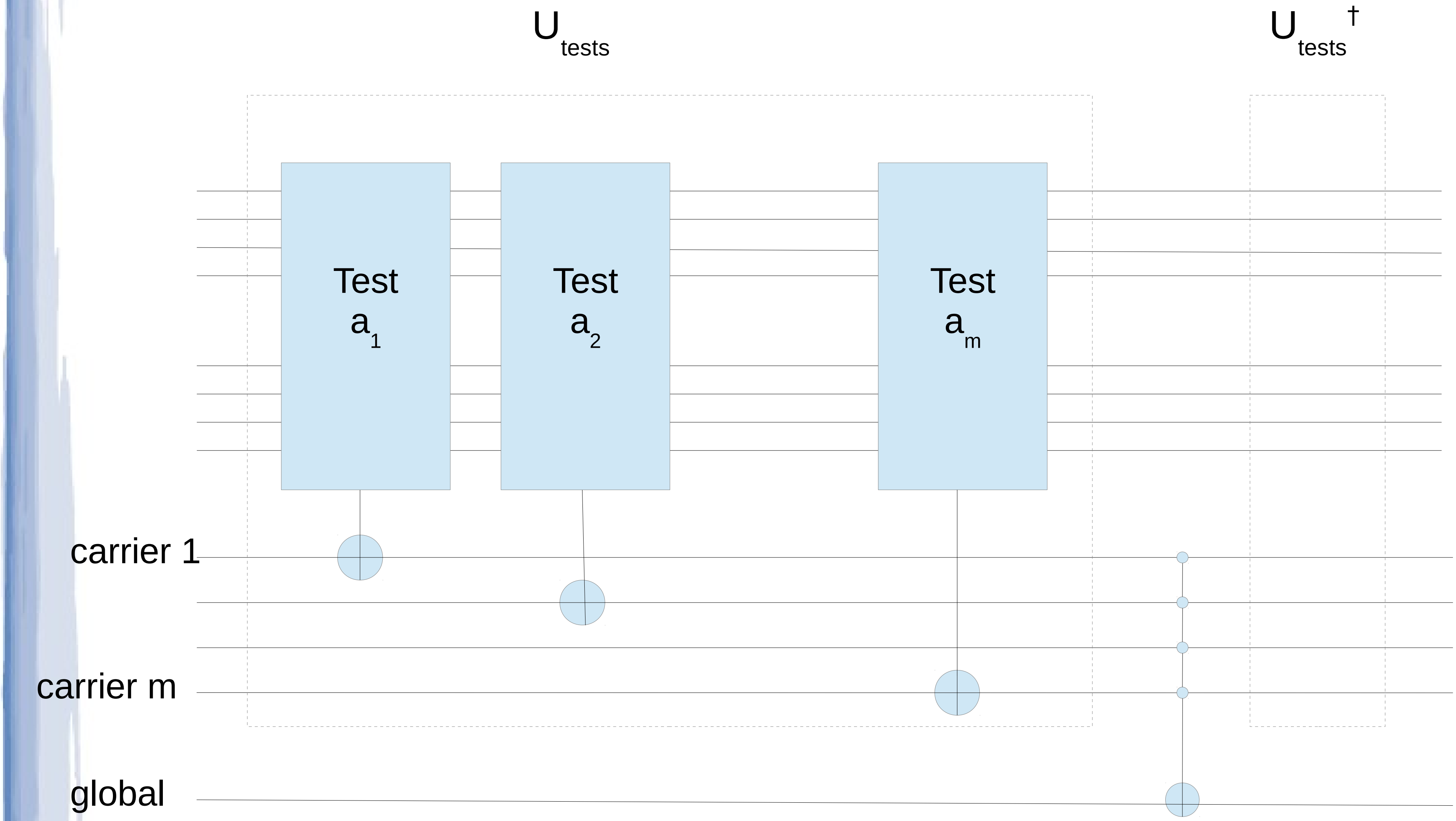
We need to construct an oracle!

Construction of a Quantum Primality oracle

An efficient Quantum Oracle can be constructed
following classical primality tests

Miller-Rabin primality test

- Find $x \rightarrow x-1=2^s d$
- Choose “witness” $1 \le a \le x$
- If $a^d \ne 1 \bmod x$ then x is composite
  $a^{2^r d} \ne -1 \bmod x \quad 0 \le r \le s-1$
- If any test fails, *x* may be prime or composite: “liars”
- Eliminate strong liars checking less than ln(*x*) witnesses

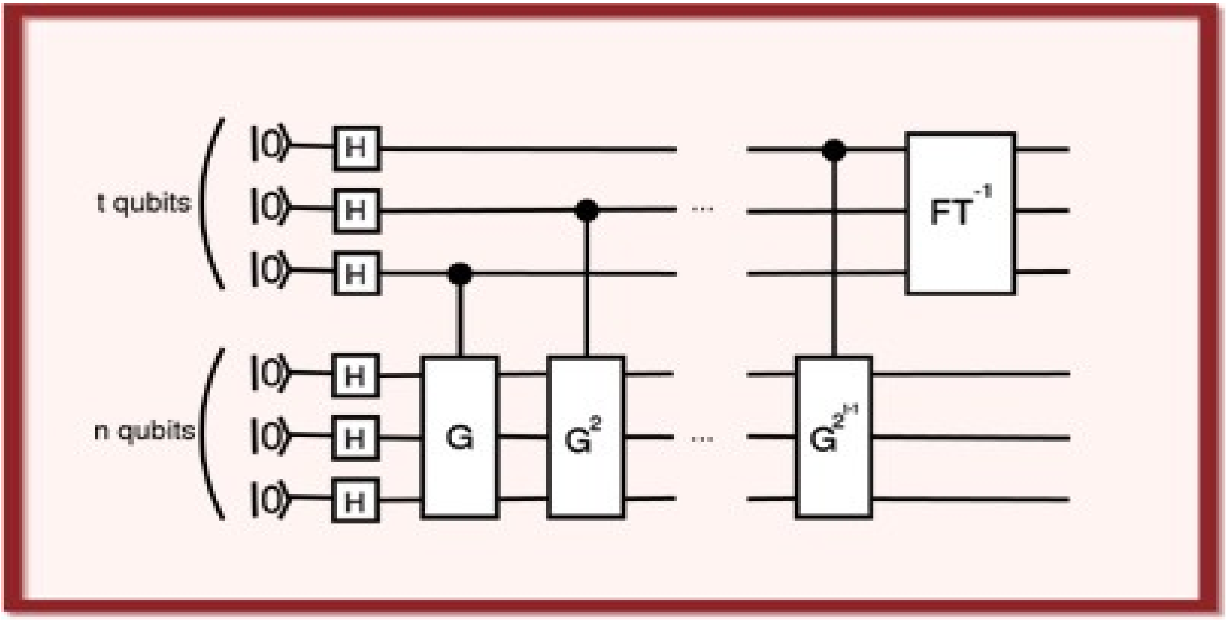


Structure of the quantum primality oracle

Quantum Counting of Prime numbers

quantum oracle + quantum Fourier transform
= quantum counting algorithm

Brassard, Hoyer, Tapp (1998)



Counts the number of solutions to the oracle

We want to count *M* solutions out of *N* possible states

We know an estimate $\tilde{M}$

$$|\tilde{M}-M|<\frac{2\pi}{c}M^{\frac{1}{2}}+\frac{\pi^2}{c^2}$$

Bounded error in quantum counting using $c\sqrt{N}$ calls

Brassard, Hoyer, Tapp (1998)

Bounded error in the quantum counting of primes

$$|Li(x)-\pi_{QM}(x)|<\frac{2\pi}{c}\frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}}x}$$

$$|Li(x) - \pi_{QM}(x)| < \frac{2\pi}{c} \frac{x^{\frac{1}{2}}}{\ln^{\frac{1}{2}} x}$$

Error of counting is smaller than the bound for the fluctuations if Riemann conjecture is correct!

Best classical algorithm by Lagarias-Miller-Odlyzko (1987)
implemented by Platt (2012)

$$T \sim x^{\frac{1}{2}} \qquad S \sim x^{\frac{1}{4}}$$

A Quantum Computer could calculate the size of fluctuations more efficiently than a classical computer

Conclusion

I) Quantum Simulation of Geometry and Topology

Geometry ↔ Site-dependent coupling

Dimensions ↔ Connectivity

Boundary conditions ↔ Non-local coupling

II) Quantum Simulation of Arithmetics

Superposition of series of numbers using appropriate q-oracles

Measurements of arithmetic functions

Verification of Riemann Hypothesis

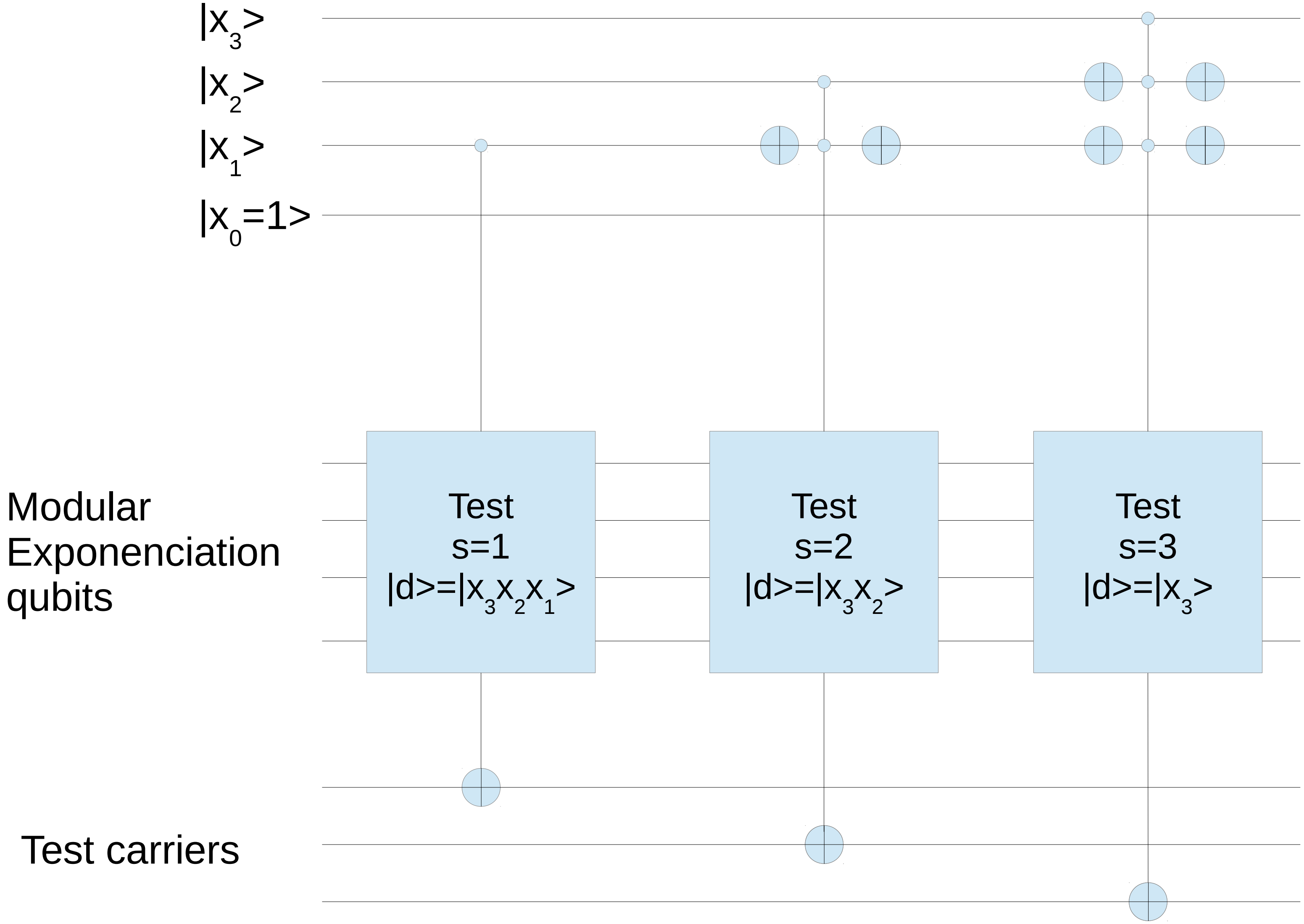
Construction of twin primes

$$U_{+2}|P(n)\rangle|0\rangle=\sum_{p\in primes}|p+2\rangle|0\rangle$$

$$U_{primality}\,U_{+2}|P(n)\rangle|0\rangle=\sum_{p,\,p+2\in primes}|p+2\rangle|0\rangle+\sum_{p+2\notin primes}|p+2\rangle|1\rangle$$

$$Prob(|twin\,primes\rangle)=\frac{\pi_2(2^n)}{\pi(2^n)}\approx\frac{1}{(n\ln 2)^2}$$

Efficient construction!



$|x_3>$

$|x_2>$

$|x_1>$

$|x_0=1>$

Modular
Exponenciation
qubits

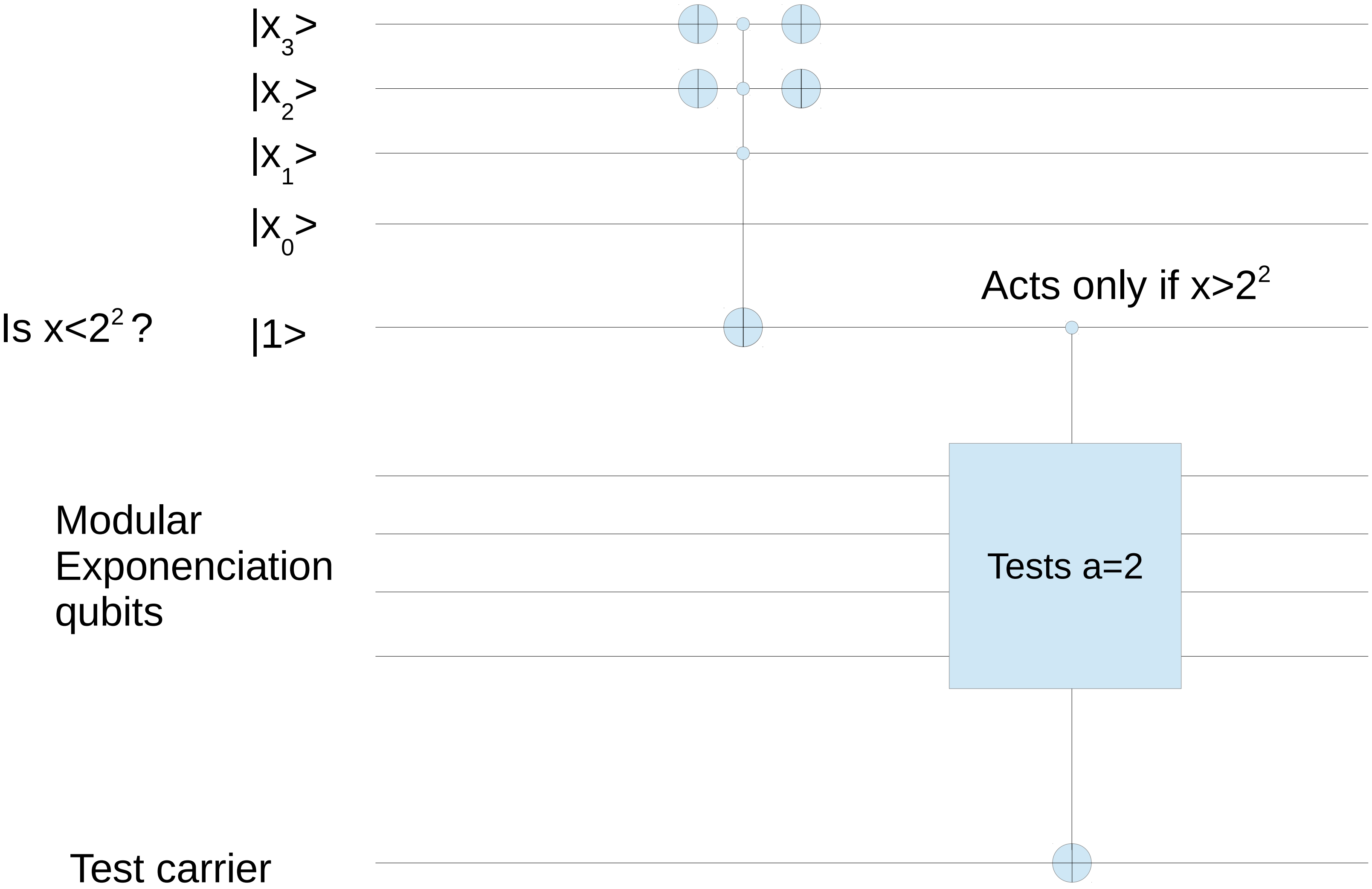
Test
s=1
$|d>=|x_3x_2x_1>$

Test
s=2
$|d>=|x_3x_2>$

Test
s=3
$|d>=|x_3>$

Test carriers

Tests are conditioned to the actual value of *x*



$|x_3>$

$|x_2>$

$|x_1>$

$|x_0>$

Acts only if $x>2^2$

Is $x<2^2$?

$|1>$

Modular
Exponenciation
qubits

Tests a=2

Test carrier

Beyond Prime numbers: the **q-functor**

$$f : S \subseteq X \rightarrow |S\rangle = \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$$

$$|S\rangle = \frac{1}{\sqrt{|S|}} \sum_{x \in X} \chi_S(x) |x\rangle \qquad \chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

Primes
Average (Cramér) primes
Dirichlet characters
...

Only needs the construction of a quantum oracle for **$\chi_s(x)$**