Note Title 9/1/2005

CLUSTER STATES

&

LINEAR OPTICS

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1/ Review of the circuit model
2/ Abstract cluster state model
3/ Linear optical Cluster States

1) REVIEW OF CIRCUIT MODEL

→ Start with 2-state quantum systems

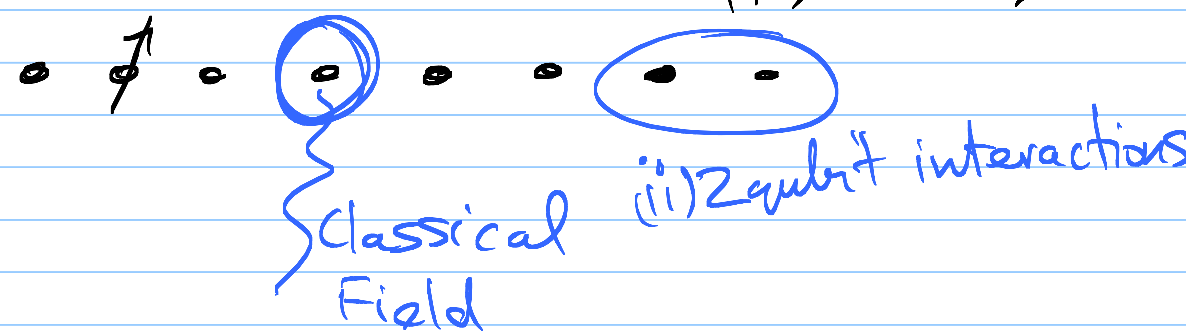
$|0\rangle \qquad |1\rangle$

$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$

e.g spin-½ particle $|\tfrac{1}{2},\tfrac{1}{2}\rangle \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle$

$|0\rangle \qquad |1\rangle$

$|\uparrow\rangle \qquad |\downarrow\rangle$



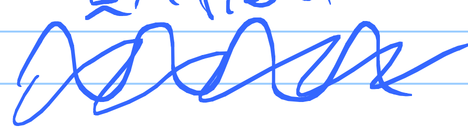
(i) Single qubit interactions:

e.g

$|e\rangle$

$|g\rangle$

Static $\underset{\sim}{E}$ or Classical EM field



$U = 2\times 2$ matrix $\quad e^{-iHt}$

(ii) 2-qubit gates

$U \to 4\times 4$ matrix

CNOT $\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$ $\quad \begin{matrix} & 1 \\ 0 & 1 \\ 1 & 0 \end{matrix}$

$C\text{-}Z = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \leftarrow \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{array}$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>Exercise</u> Show that if I write the

C-Z in this basis

$$|0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle$$

where $|\pm\rangle = |0\rangle \pm |1\rangle$

(eigenstates of X)

then

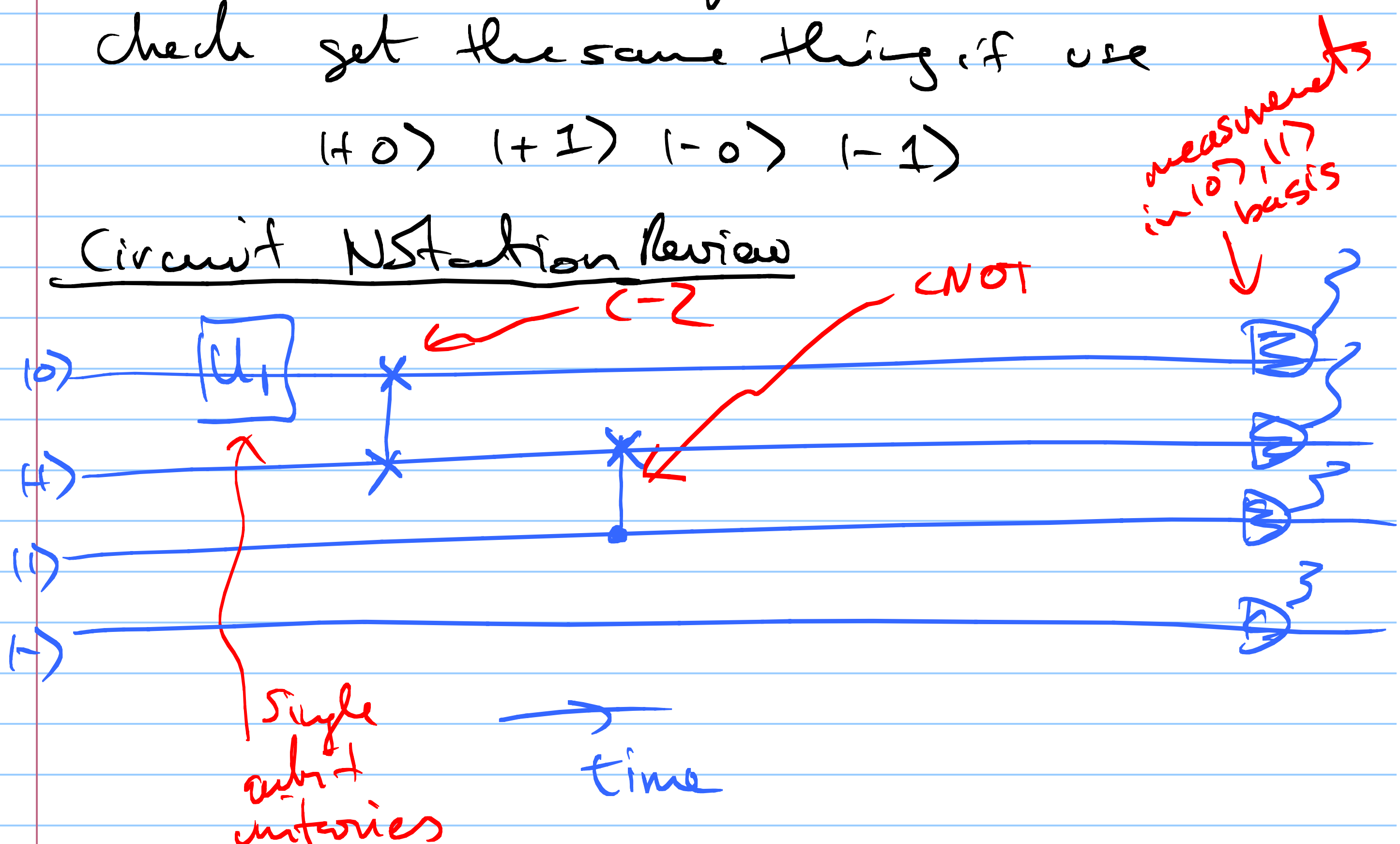
$$C\text{-}Z = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$$

"flips second qubit if first qubit is in the state 1"

check get the same thing if use

$$|+0\rangle \quad |+1\rangle \quad |-0\rangle \quad |-1\rangle$$

Circuit Notation Review



Universality: single qubit gates + C-Z gates allow us to build all unitaries

Big Picture:

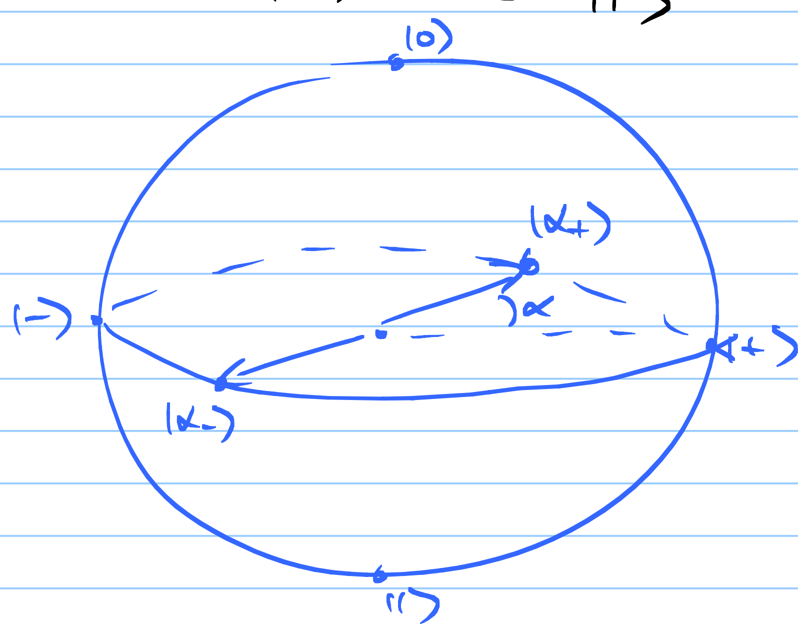
$$\boxed{\text{Product State}} \Longrightarrow \boxed{\Psi_{\text{entangled}}} \overset{\text{si}}{\Longrightarrow} \boxed{\text{Single qubit measurements}}$$

Final notational stuff

$|0\rangle\ |1\rangle$ eigenstates of Z

$|+\rangle\ |-\rangle$ " " X

$$|\alpha_\pm\rangle \equiv e^{i\alpha/2}|0\rangle \pm e^{-i\alpha/2}|1\rangle$$



Other Useful gates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Z_\alpha = e^{-i\alpha/2\, \overset{\sigma_z}{Z}} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \qquad X_\beta = \begin{pmatrix} \cos\beta/2 & -i\sin\beta/2 \\ -i\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

ie subscripts indicate a rotation, not a pauli matrix!

SECTION 2 CLUSTER COMPUTATION

(Raussendorf & Briegel)

Begin with entangled state ⇒ Single qubit measurements the nature of which determine the algorithm we're running

⇓

Final set of computational basis measurements

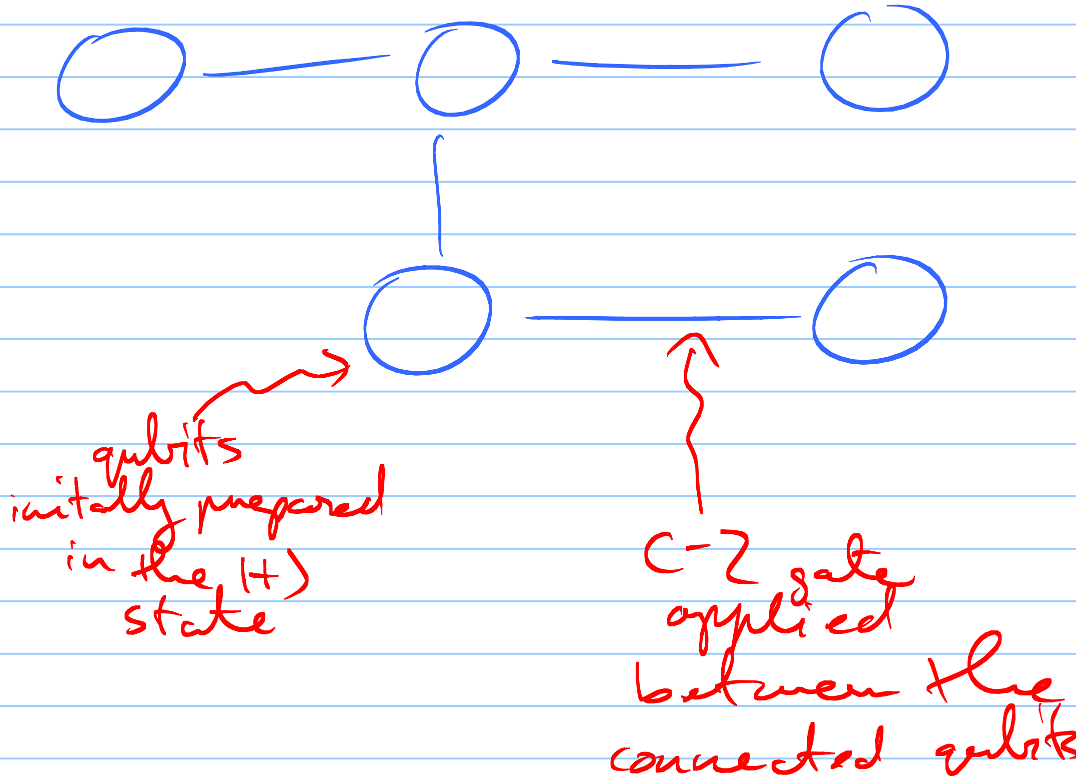
t

BIG PICTURE:

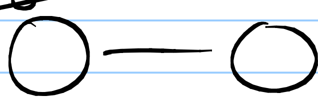
- If the circuit model required n qubits & L steps (gate operations) then the cluster state model requires $\sim n \times L$ qubits in a special initial "cluster" entangled state.

What is the initial entangled state?

Graphical representation



e.g. 1



$$(|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle) = |00\rangle+|01\rangle+|10\rangle+|11\rangle$$

$$\Big\downarrow \text{C-Z}$$

$$|00\rangle+|01\rangle+|10\rangle-|11\rangle$$

equivalent to:

$$|0\rangle|+\rangle+|1\rangle|-\rangle \qquad\qquad |+0\rangle+|-1\rangle$$

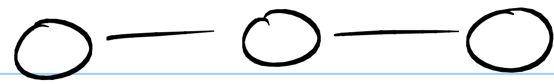
$$|+\rangle|+\rangle = (|0\rangle+|1\rangle)|+\rangle$$

(Recall Exercise)

$$\Big\Downarrow \text{C-Z}$$

$$|0\rangle|+\rangle+|1\rangle|-\rangle$$

e.g 2



$$|+\rangle \qquad (|0\rangle + |1\rangle) \qquad |+\rangle$$

$$= |+\rangle|0\rangle|+\rangle + |+\rangle|1\rangle|+\rangle$$

$$= |+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle = |L_3\rangle$$

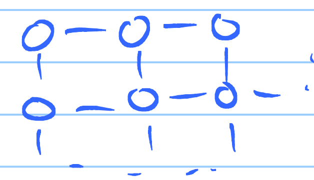
$$(|0\rangle + |1\rangle)\,|+\rangle\,(|0\rangle + |1\rangle)$$

$$\begin{aligned} &|0 \; + \; 0\rangle \\ + \; &|0 \; - \; 1\rangle \\ + \; &|1 \; - \; 0\rangle \\ + \; &|1 \; + \; 1\rangle \end{aligned} \equiv |L_3\rangle$$

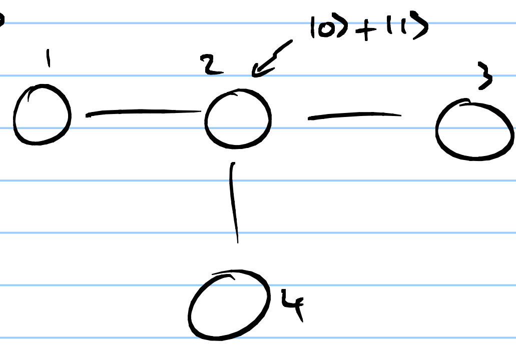
An aside question re "graph" versus "cluster" states: Q1

crazy people use

"Cluster State" only for



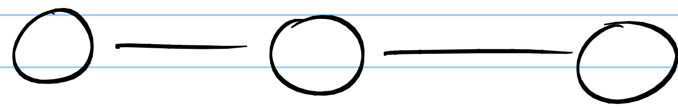
e.g 3



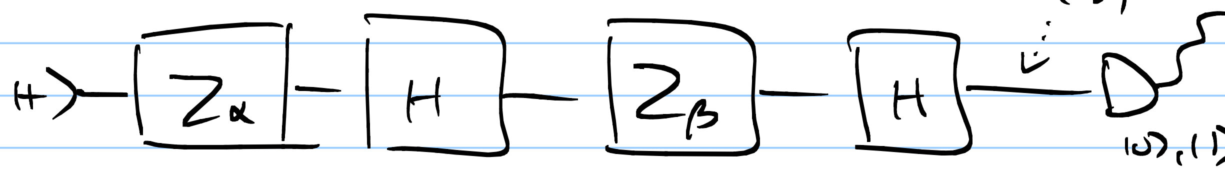
$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ & |+ & 0 & + & +\rangle \\ + & |- & 1 & - & -\rangle \end{array}$$

Lecture 2

How to use a cluster state?

Use  $= |L_3\rangle$

to simulate   state here we call $\begin{pmatrix} A \\ B \end{pmatrix}$



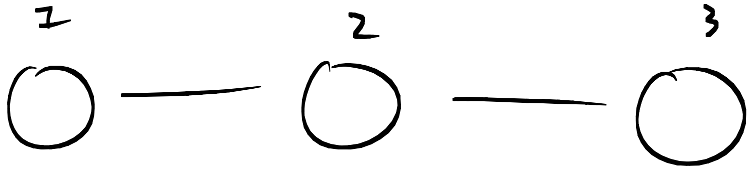
~~2~~ Output of the circuit

$$H Z_\beta H = X_\beta$$

$$\begin{pmatrix} \cos\beta/2 & -i\sin\beta/2 \\ -i\sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$$

$$\begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} \\ e^{i\alpha/2} \end{pmatrix}$$

$$\begin{pmatrix} \cos\beta/2 \, e^{-i\alpha/2} - i\sin\beta/2 \, e^{i\alpha/2} \\ -i\sin\beta/2 \, e^{-i\alpha/2} + \cos\beta/2 \, e^{i\alpha/2} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$



(i) Measure qubit 1 in

$$\{ |\alpha_+\rangle\langle\alpha_+| \,;\, |\alpha_-\rangle\langle\alpha_-| \}$$

$$|\alpha_\pm\rangle = e^{i\alpha/2}|0\rangle \pm e^{-i\alpha/2}|1\rangle$$

(Exercise) $= \cos\alpha/2\,|\pm\rangle + i\sin\alpha/2\,|\mp\rangle$ $\quad \{ \langle\alpha_\pm| = \cos\alpha/2\,\langle\pm| - i\sin\alpha/2\,\langle\mp|$

Say get $|\alpha_+\rangle\langle\alpha_+|$ outcome:

$$\langle\alpha_+|\cdot|L_3\rangle = \langle\alpha_+|\big(|+0+\rangle + |-1-\rangle \big)$$

$$= \cos\alpha/2\,|0+\rangle - i\sin\alpha/2\,|1-\rangle$$

(ii) Measure qubit 2 in $\{|\beta_+\rangle\langle\beta_+|, |\beta_-\rangle\langle\beta_-|\}$

$\langle\beta_+| : e^{-i\beta/2}\langle 0| + e^{i\beta/2}\langle 1| :$

$$\cos\alpha/2\,|0+\rangle - i\sin\alpha/2\,|1-\rangle$$

$$\Longrightarrow e^{-i\beta/2}\cos\alpha/2\,|+\rangle - i\sin\alpha/2\, e^{i\beta/2}\,|-\rangle$$

$$\left(e^{-i\beta/2}\cos\alpha/2 - i\sin\alpha/2\, e^{i\beta/2}\right)|0\rangle$$

$$+ \left(e^{-i\beta/2}\cos\alpha/2 + i\sin\alpha/2\, e^{i\beta/2}\right)|1\rangle$$

$$= \left(\cos\beta/2\, e^{-i\alpha/2} - i\sin\beta/2\, e^{+i\alpha/2}\right)|0\rangle$$

$$+ \left(\cos\beta/2\, e^{i\alpha/2} + i\sin\beta/2\, e^{-i\alpha/2}\right)|1\rangle$$

$$\begin{pmatrix} \cos\beta/2\, e^{-i\alpha/2} - i\sin\beta/2\, e^{i\alpha/2} \\ -i\sin\beta/2\, e^{-i\alpha/2} + \cos\beta/2\, e^{i\alpha/2} \end{pmatrix} \equiv \begin{pmatrix} A \\ B \end{pmatrix}$$

$\therefore$ With probability $\frac{1}{2}\times\frac{1}{2}$ we get the same as the circuit output.

(Ex: Prove an $\{|\alpha_+\rangle\langle\alpha_+|, |\alpha_-\rangle\langle\alpha_-|\}$ on ANY cluster qubit has probability $1/2$)

What if got $|\alpha_+\rangle\langle\alpha_+|$ but then got $|\beta_-\rangle\langle\beta_-|$

$$\langle\beta_\pm| : e^{-i\beta/2}\langle 0| \pm e^{i\beta/2}\langle 1| : \cos\alpha/2\, |0+\rangle - i\sin\alpha/2\, |1-\rangle$$

$$\Rightarrow e^{-i\beta/2}\cos\alpha/2\, |+\rangle \mp i\sin\alpha/2\, e^{i\beta/2}\, |-\rangle$$

$$\left(e^{-i\beta/2}\cos\alpha/2 \mp i\sin\alpha/2\, e^{i\beta/2}\right)|0\rangle$$
$$+ \left(e^{-i\beta/2}\cos\alpha/2 \pm i\sin\alpha/2\, e^{i\beta/2}\right)|1\rangle$$

$$= \left(\cos\beta/2\, e^{\mp i\alpha/2} - i\sin\beta/2\, e^{\pm i\alpha/2}\right)|0\rangle$$
$$+ \left(\cos\beta/2\, e^{-i\alpha/2} - i\sin\beta/2\, e^{\pm i\alpha/2}\right)|1\rangle \quad \equiv \begin{pmatrix} B \\ A \end{pmatrix}$$

$$\begin{pmatrix} \cos\beta/2\, e^{-i\alpha/2} - i\sin\beta/2\, e^{i\alpha/2} \\ -i\sin\beta/2\, e^{-i\alpha/2} + \cos\beta/2\, e^{i\alpha/2} \end{pmatrix} \equiv \begin{pmatrix} A \\ B \end{pmatrix}$$

Do mental readjustment on final $|0\rangle, |1\rangle$ basis measurement

What if got $\langle\alpha_-|$ outcome initially?

$\langle\alpha_-|L_3\rangle$

$$= -i\sin\alpha/2\,|0+\rangle + \cos\alpha/2\,|1-\rangle$$

But we wanted: $\quad$ " $\quad |1-\rangle \quad$ " $\quad |0+\rangle \quad$ (if $\langle\alpha_+|$ outcome)

Could apply $X^{(2)}Z^{(3)}$ ⇐

Apply $X^{(2)}$ to the $|\beta_\pm\rangle\langle\beta_\pm|$ measurement instead

$$|\beta_\pm\rangle = e^{i\beta/2}|0\rangle \pm e^{-i\beta/2}|1\rangle$$

$$X|\beta_\pm\rangle = e^{i\beta/2}|1\rangle \pm e^{-i\beta/2}|0\rangle$$

$$X|\beta_+\rangle\langle\beta_+|X = |(-\beta)_+\rangle\langle(-\beta)_+|$$

$$X|\beta_-\rangle\langle\beta_-|X = |(-\beta)_-\rangle\langle(-\beta)_-|$$

This is the key idea: classically feed forward & change the angle of the second measurement based on the outcome of the first one.

Exercise: Check that the $|\beta_-\rangle\langle\beta_-|$ outcome is also fine

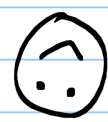
In summary:

(i) $\{|\alpha_+\rangle\langle\alpha_+|,\ |\alpha_-\rangle\langle\alpha_-|\}$

classical feedforward to switch the angle from $+\beta \to -\beta$

$\{|\beta_+\rangle\langle\beta_+|;\ |\beta_-\rangle\langle\beta_-|\}$ $\qquad$ $\{|-\beta_+\rangle\langle-\beta_+|,\ |-\beta_-\rangle\langle-\beta_-|\}$

$\begin{pmatrix} A \\ B \end{pmatrix}$ $\qquad$ $\begin{pmatrix} B \\ A \end{pmatrix}$ $\qquad$ $\begin{pmatrix} A \\ -B \end{pmatrix}$ $\qquad$ $\begin{pmatrix} -B \\ A \end{pmatrix}$

 $\qquad$ 

(X error → mental readjustment)

(Fine if this is the output qubit)

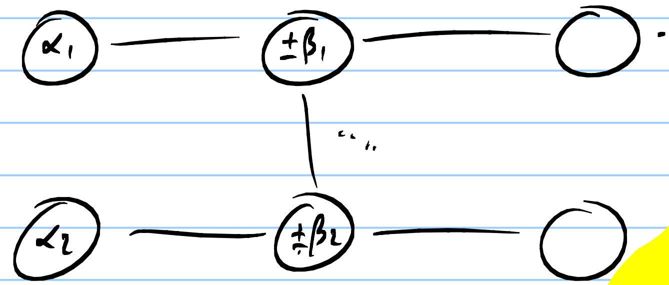
Mental readjustment again

If you understand this example you are a long way toward understanding all cluster computations. They all proceed via similar measurements whose angle may depend on the particular previous outcomes obtained. These "Pauli errors" (extra X & Z gates which have been apparently "applied" to the qubits) push through the computation, causing angles to be adjusted, & possibly a final 'mental readjustment'

Exercise

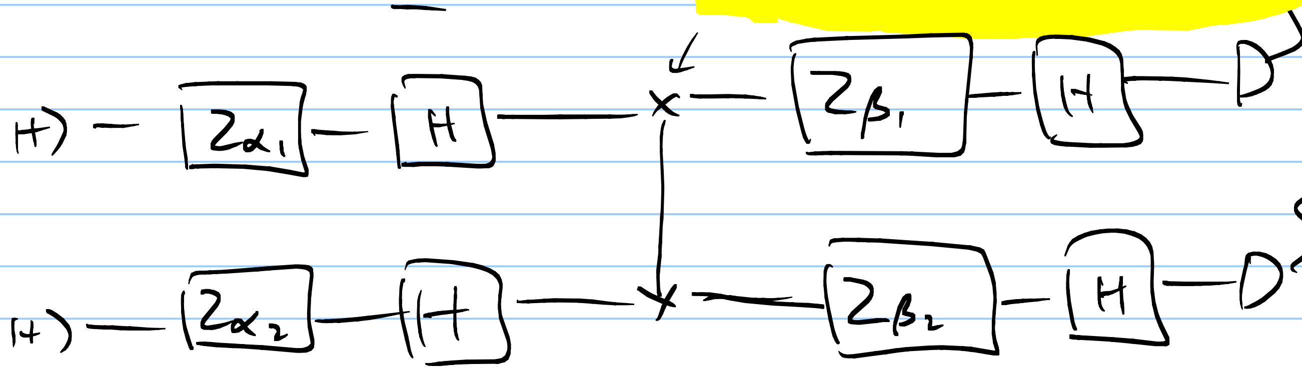
depend on the previous outcomes

More complicated circuits

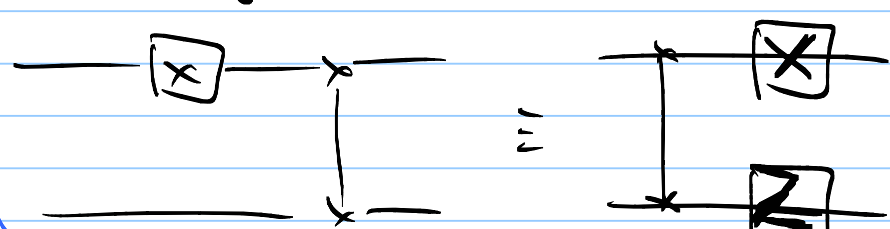


Ultimately ALL circuits are of similar form, & thus can be simulated within the cluster model

=

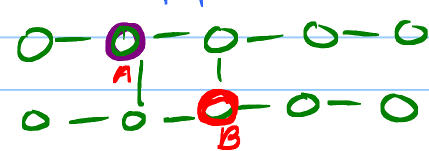


Note: Pauli errors pushing through C-Z gate, fan out in a light cone



This means that cluster measurements may depend on outcomes in different "rows":

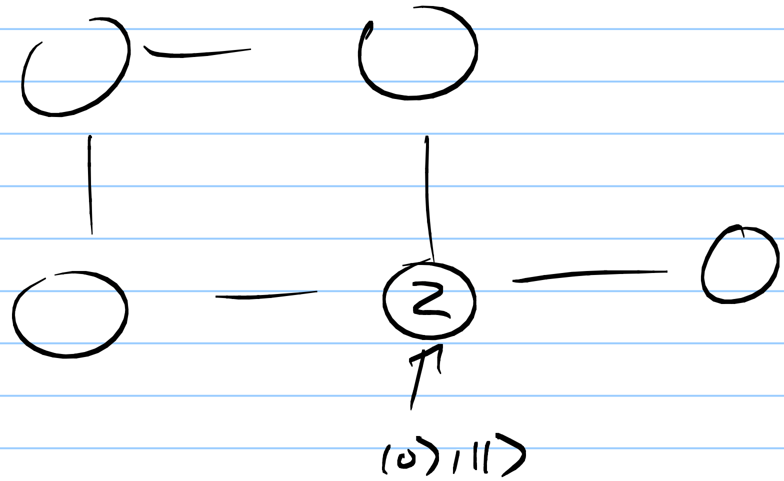
For example



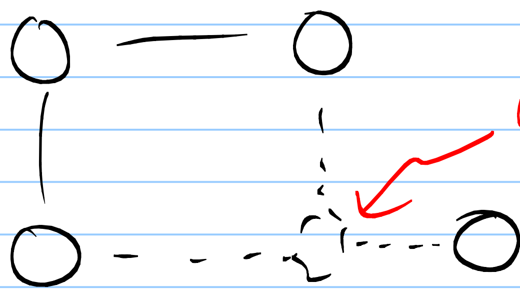
Angle at which qubit B is measured may depend on outcome at qubit A

Some properties of generic clusters

(i)



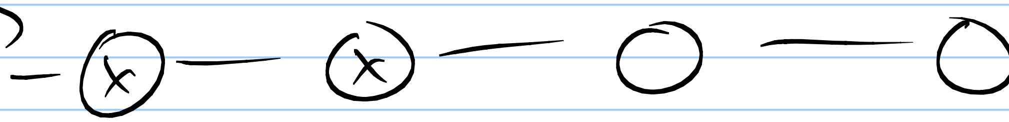
Z measurement breaks the qubit out of the cluster



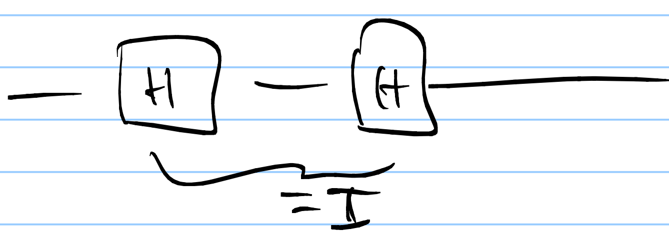
cluster qubit gets removed & all its bonds severed. This may cause Pauli errors on the adjacent qubits, but such are easily compensated for as we have seen

(ii) X measurements propagate states

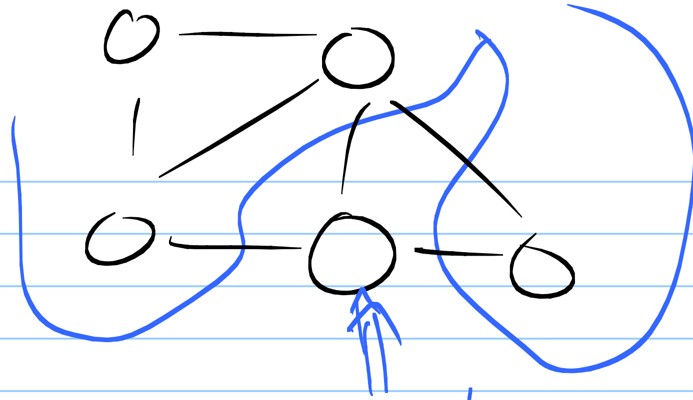
(4)



$|\pm\rangle \equiv \alpha = 0 \quad |\alpha_\pm\rangle$



(iii)



Bi-orthogonal decomposition for any "singled out" cluster qubit.

$$|A\rangle|0\rangle + |A^{\perp}\rangle|1\rangle$$

rest of cluster

singled out qubit.

SECTION 3 PHOTONIC CLUSTERS

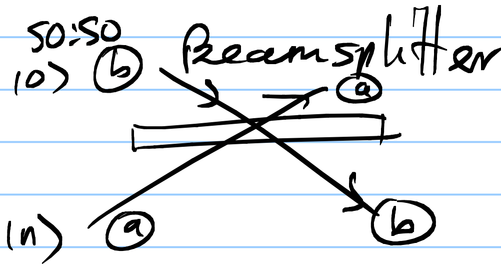
Linear Optical Elements Intro

$|n_\lambda\rangle$ n photons in mode λ → orbital ang. momentum

→ time

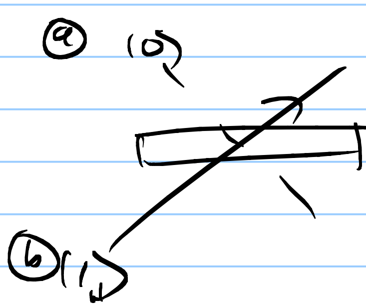
→ spatial mode

→ Polarization



$$a_H^\dagger \to (a_H^\dagger + b_H^\dagger)/\sqrt{2}$$

$$b_H^\dagger \to (a_H^\dagger - {}^\dagger_H)/\sqrt{2}$$



You "know"

$|01\rangle \pm |10\rangle$

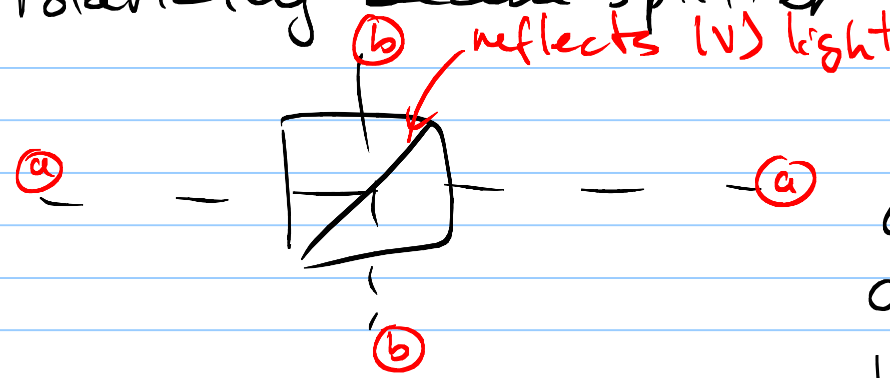
How to compute this?

$$|0\rangle_a |1\rangle_b = b_H^\dagger |0\rangle$$

evolves to $\longrightarrow$ $(a_H^\dagger - b_H^\dagger)|0\rangle$

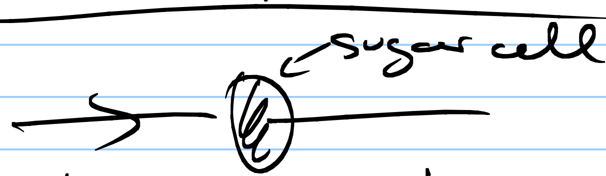
$$= |01\rangle - |10\rangle$$

Polarizing Beam Splitter



$$a_H^\dagger \to a_H^\dagger$$
$$a_V^\dagger \to b_V^\dagger$$
$$b_H^\dagger \to b_H^\dagger$$
$$b_V^\dagger \to a_V^\dagger$$

Polarization Rotator



$$a_H^\dagger \to \cos\theta\, a_H^\dagger + \sin\theta\, a_V^\dagger, \quad a_V^\dagger \to \sin\theta\, a_H^\dagger - \cos\theta\, a_V^\dagger$$

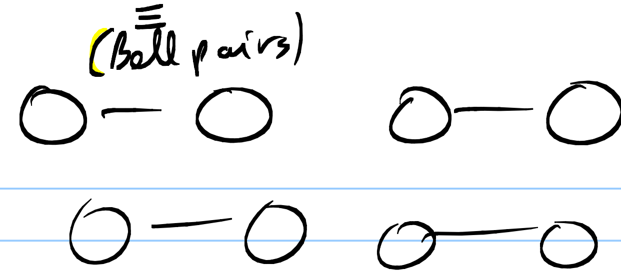
USING LINEAR OPTICS TO NON DETERMINISTICALLY "FUSE" PHOTONIC CLUSTERS TOGETHER:

→ Hard to get individual photons to interact, especially in a deterministic manner.

→ But with clusters the interactions that cause the entanglement can be done 'a-priori' (offline), so that if they screw up they are not ruining the computation per se.

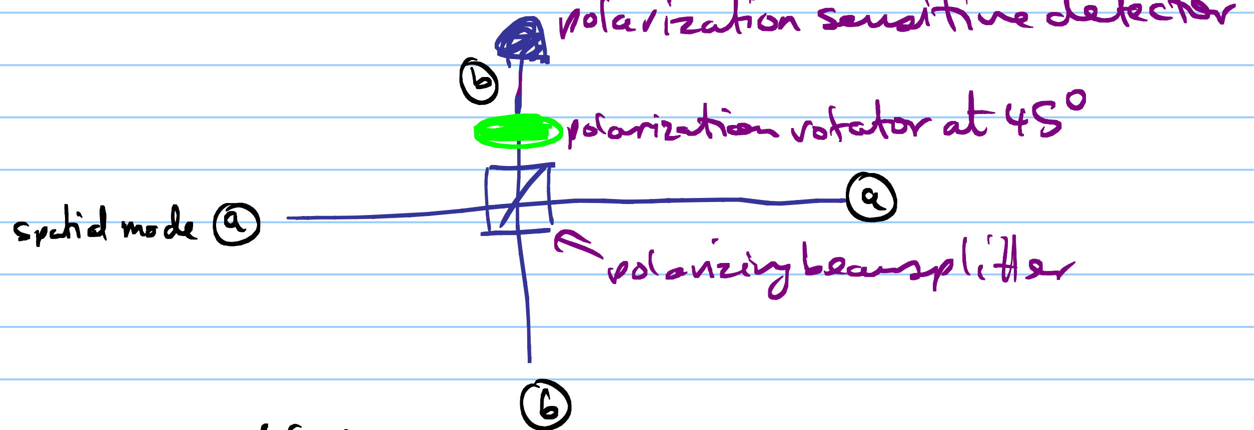
Lets see how with a simple example:

Imagine I have a resource of small, 2qubit clusters.



How to join them to make larger clusters?

Consider the following very simple linear optical gate:



Work out the effect of the gate on four possible input pairs:

(i) $|H\rangle_a |H\rangle_b \xrightarrow{PBS} |H\rangle_a |H\rangle_b \xrightarrow{Rotator} |H\rangle_a (|H\rangle_b + |V\rangle_b)$

(ii) $|V\rangle_a |V\rangle_b \xrightarrow{PBS} |V\rangle_b |V\rangle_a \xrightarrow{Rotator} (|H\rangle_b - |V\rangle_b) |H\rangle_a$

(iii) $|H\rangle_a |V\rangle_b \xrightarrow{PBS} |H\rangle_a |V\rangle_a \xrightarrow{Rotator} |H\rangle_a |V\rangle_a$

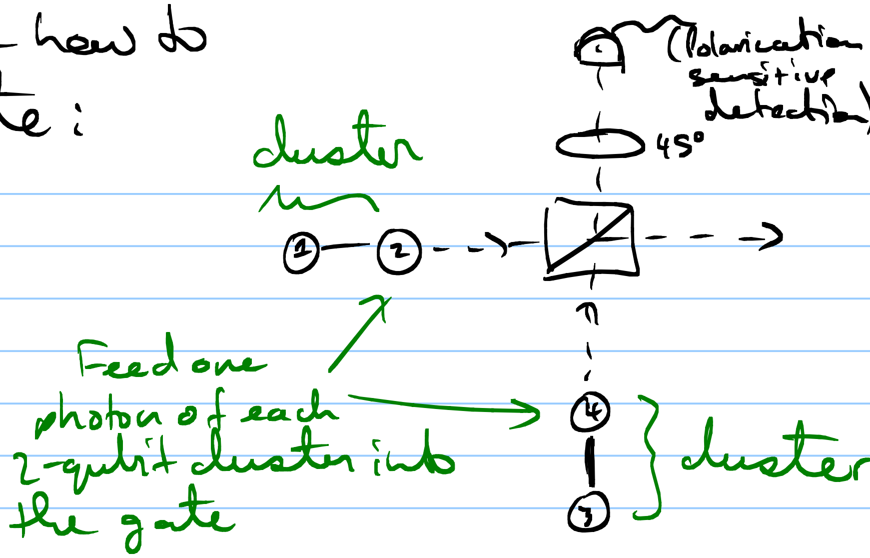
(iv) $|V\rangle_a |H\rangle_b \longrightarrow |V\rangle_b |H\rangle_b \rightarrow (|H\rangle_b - |V\rangle_b)(|H\rangle_b + |V\rangle_b)$

$= |2_H\rangle_b - |2_V\rangle_b$

This means 2-horizontal photons in spatial mode b

Key feature: Only the HaHb or VaVb inputs can lead to a single photon being detected. Crucially when such a photon is detected you don't learn whether the photons were both "H" or both "V", only that they were "the same": A measurement of "even parity"

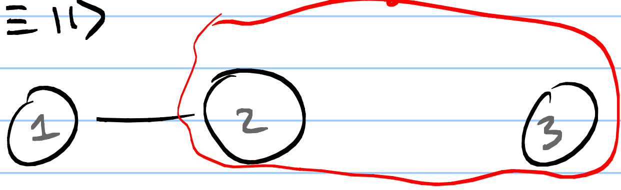
Now lets see how to use this gate:



$\Rightarrow$ Say the gate succeeds (ie a single photon is detected).

$|H\rangle \equiv |0\rangle$
$|V\rangle \equiv |1\rangle$

Apply the gate on these two photons



$|+\rangle|H\rangle + |-\rangle|V\rangle \qquad |H\rangle|+\rangle + |V\rangle|-\rangle$ (4 terms)

Gate picks out these two terms, which have even parity, ie they have the same polarization

$\therefore$ Collapse to:

single photon left after the gate has succeeded

$$\underset{1}{|+\rangle}\underset{2}{|H\rangle}\underset{4}{|+\rangle} + \underset{1}{|-\rangle}\underset{2}{|V\rangle}\underset{4}{|-\rangle}$$

$$= |L_3\rangle =$$ 

So, we have "fused" qubits 2 & 3, losing one photon in the process, & created a larger cluster state.

~~SU~~ SUMMARY

Hopefully you have some inkling of how cluster state computation works now. To really understand it, you need to work thru some slightly more complicated examples. Then you should re-work through things in the STABILIZER FORMALISM.

Some useful references:

[1] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188-5191 (2001).
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