Majorana fermions: a new computational paradigm

Peter Finch
James De Lisle
Gian Palumbo
JKP

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF LEEDS

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Motivation

Majorana fermions: **building blocks** of exciting phases of matter:

- Quantum wires (1 dim)
- Kitaev's honeycomb lattice (2 dims)
- What happens in 3 dims?

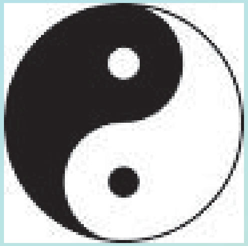
3D topological superconductor: *a new phase of matter*

Surface properties are **robust to temperature**

-> Quantum computation applications

Introduction

Majorana fermions (Majies):



Fermions which **are their own anti-particles**

$$\gamma\gamma^\dagger + \gamma^\dagger\gamma = 2 \qquad \gamma^\dagger = \gamma \qquad \gamma^2 = 1$$

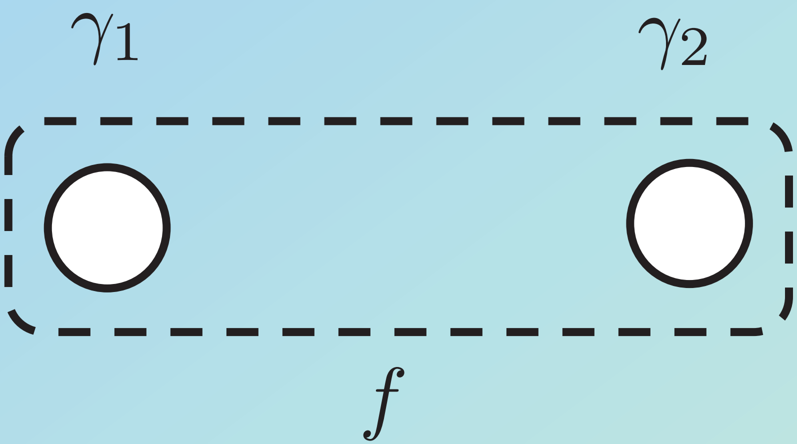
They are encountered in:

- **high energy physics**
- **condensed matter**

In 2 dims they are **also non-Abelian anyons** (Ising)

Majies: 0 Dim

2 Majies = 1 normal fermion



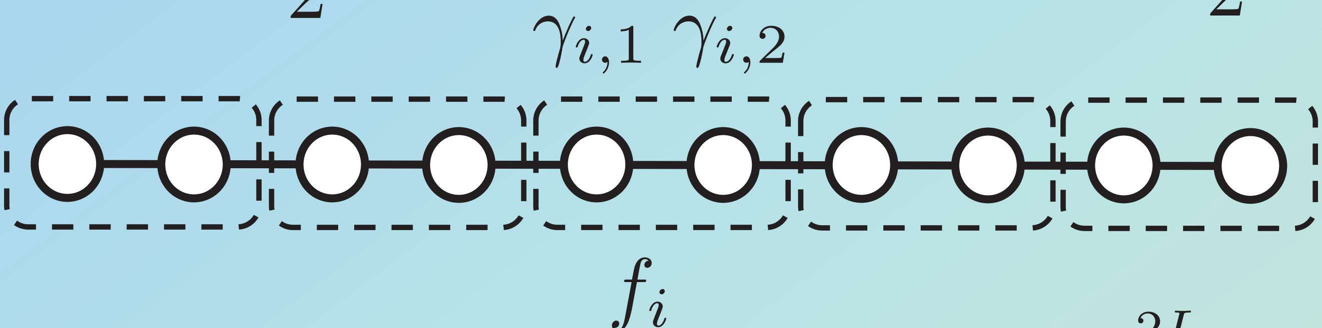
$$f = \frac{\gamma_1 + i\gamma_2}{2} \qquad f^\dagger = \frac{\gamma_1 - i\gamma_2}{2}$$

Fermionic mode occupation $\quad f^\dagger f = 0, 1$

Majies: 1 Dim

Even number of Majies: quantum wire

$$f_i = \frac{\gamma_{i,1} + i\gamma_{i,2}}{2} \qquad f_i^\dagger = \frac{\gamma_{i,1} - i\gamma_{i,2}}{2}$$



Hamiltonian of superconductor:

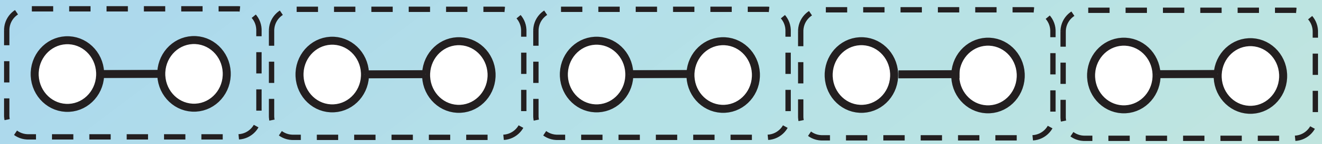
$$H = \sum_{k=1}^{2L} t_k \gamma_k \gamma_{k+1}$$

$$H = \sum_{j=1}^{L} \left[-w \left(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j \right) - \mu \left(f_j^\dagger f_j - 1/2 \right) + \left(\Delta f_j f_{j+1} + \Delta^* f_{j+1}^\dagger f_j^\dagger \right) \right]$$

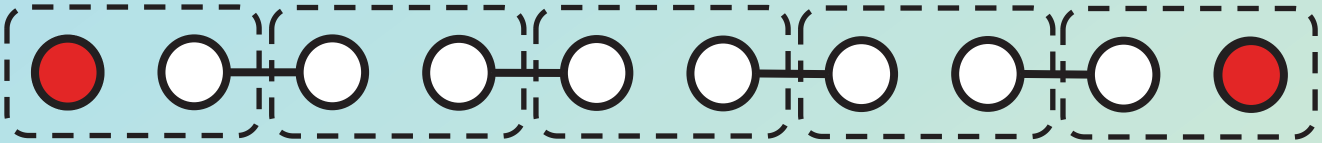
Majies: 1 Dim

- If $w = \Delta = 0$

$$H = -\frac{i\mu}{2}\sum_{i=1}^{L}\gamma_{i,1}\gamma_{i,2} = -\mu\sum_{i=1}^{L}(f_i^\dagger f_i - \frac{1}{2})$$



- If $w = \Delta,\ \mu = 0$ $\qquad H = iw\sum_{i=1}^{L-1}\gamma_{i,2}\gamma_{i+1,1}$

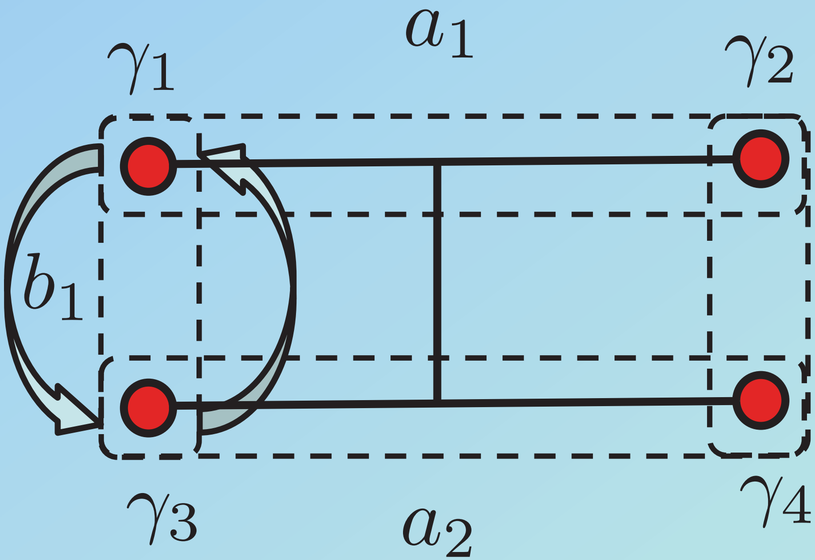


Majies appear at the edge of the wire

Majies: 1 Dim

Majorana fermions statistics:

$$\{a_1, b_1\} = \frac{1}{2}$$



Fusion

$$|11\rangle_a = \frac{1}{\sqrt{2}}(|00\rangle_b + |11\rangle_b)$$

$$|00\rangle_a = \frac{1}{\sqrt{2}}(|00\rangle_b - |11\rangle_b)$$

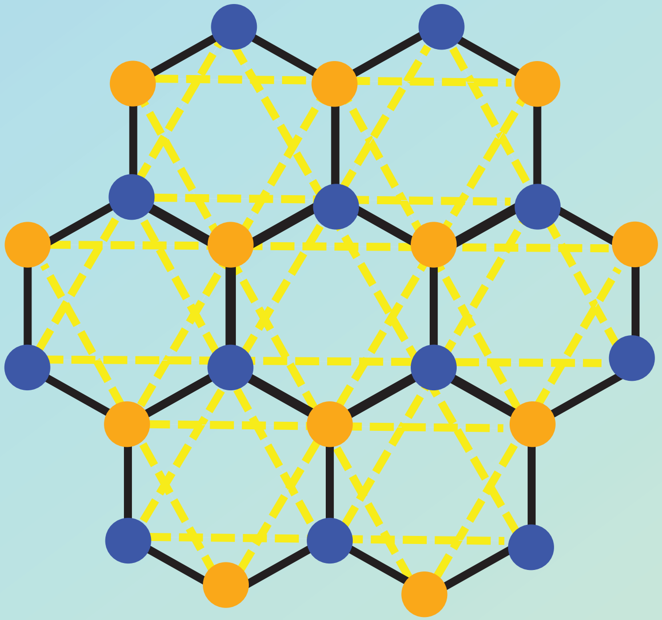
Braiding

$$\mathcal{U} = a\mathbf{1} + b\gamma_1 + c\gamma_3 + d\gamma_1\gamma_3$$

$$\mathcal{U}^2 = e^{i\pi/4}\gamma_1\gamma_3 = e^{i\pi/4}(a_1a_2 + a_1a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)$$

Majies: 2 Dim

Kitaev's honeycomb lattice



$$H = \pm i \sum_{\langle i,j \rangle} \gamma_i \gamma_j$$

- Analytically tractable
- Topological character: winding number

$$T^2 \to S^2$$

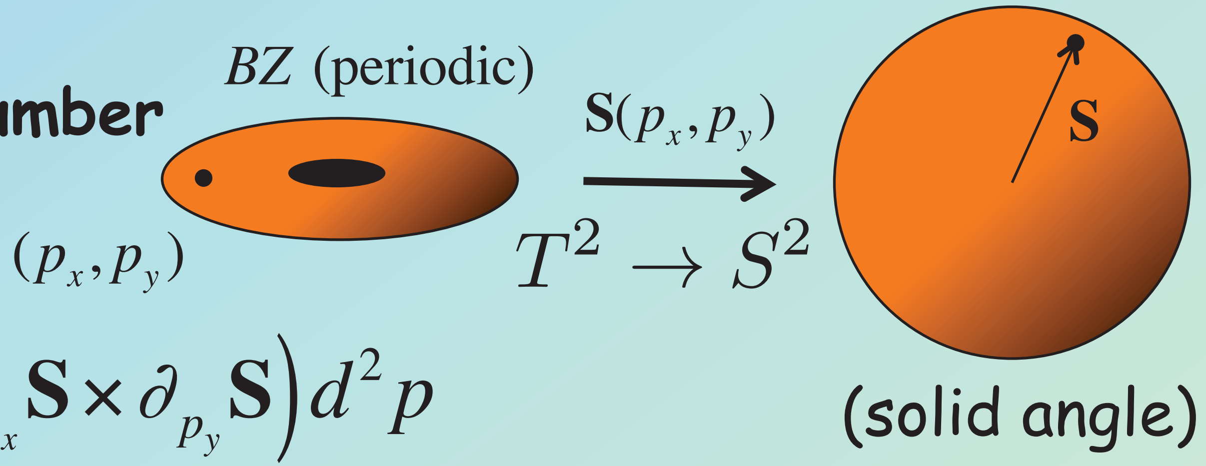
- It supports **vortices** that behave like **Majorana fermions** (same as in 1 Dim)

Majies: 2 Dim

Kitaev's honeycomb lattice

$$H = \sum_{\mathbf{p}} (f^\dagger_{\mathbf{p}}\; f_{-\mathbf{p}}) h(\mathbf{p}) \begin{pmatrix} f_{\mathbf{p}} \\ f^\dagger_{-\mathbf{p}} \end{pmatrix} \qquad h(\mathbf{p}) = E(\mathbf{p})\mathbf{S}(\mathbf{p}) \cdot \boldsymbol{\sigma}$$

- **Chern (winding) number**

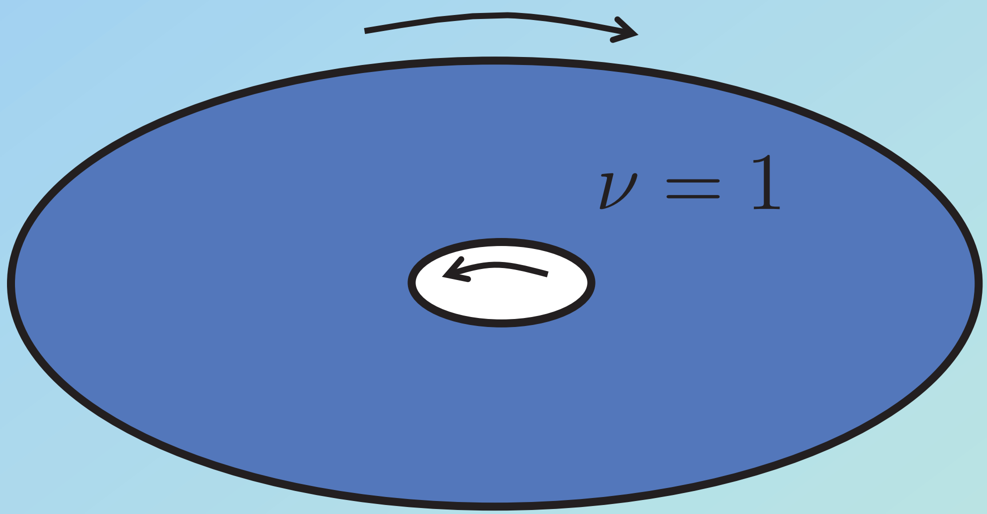


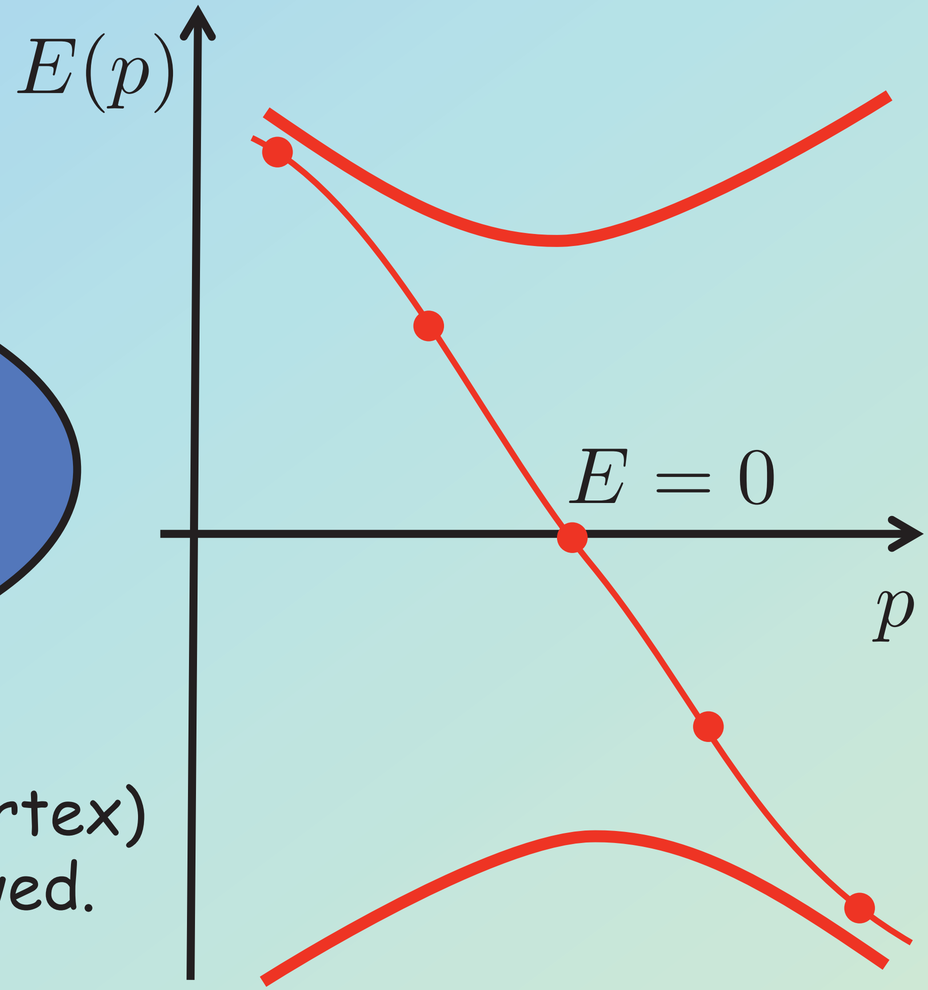
$$\nu = \frac{1}{4\pi} \int_{BZ} \mathbf{S} \cdot \left(\partial_{p_x} \mathbf{S} \times \partial_{p_y} \mathbf{S} \right) d^2 p$$

- **Particle-hole symmetry**: if $\Psi^\dagger_E$ is stationary state then Ψ_{-E} is also stationary state.

Majies: 2 Dim

Kitaev's honeycomb lattice





If π-flux through the hole (vortex) then zero energy mode is allowed.

Then $\Psi^{+}_{E=0} = \Psi_{-E=0}$ is single Majorana zero mode localised at vortex.

Majies: 3 Dim

3D:

- Can we build 3D topological superconductor out of interacting Majorana fermions?

- What is topological order?

- What are the edge (surface) states?

- Majorana fermions anywhere?

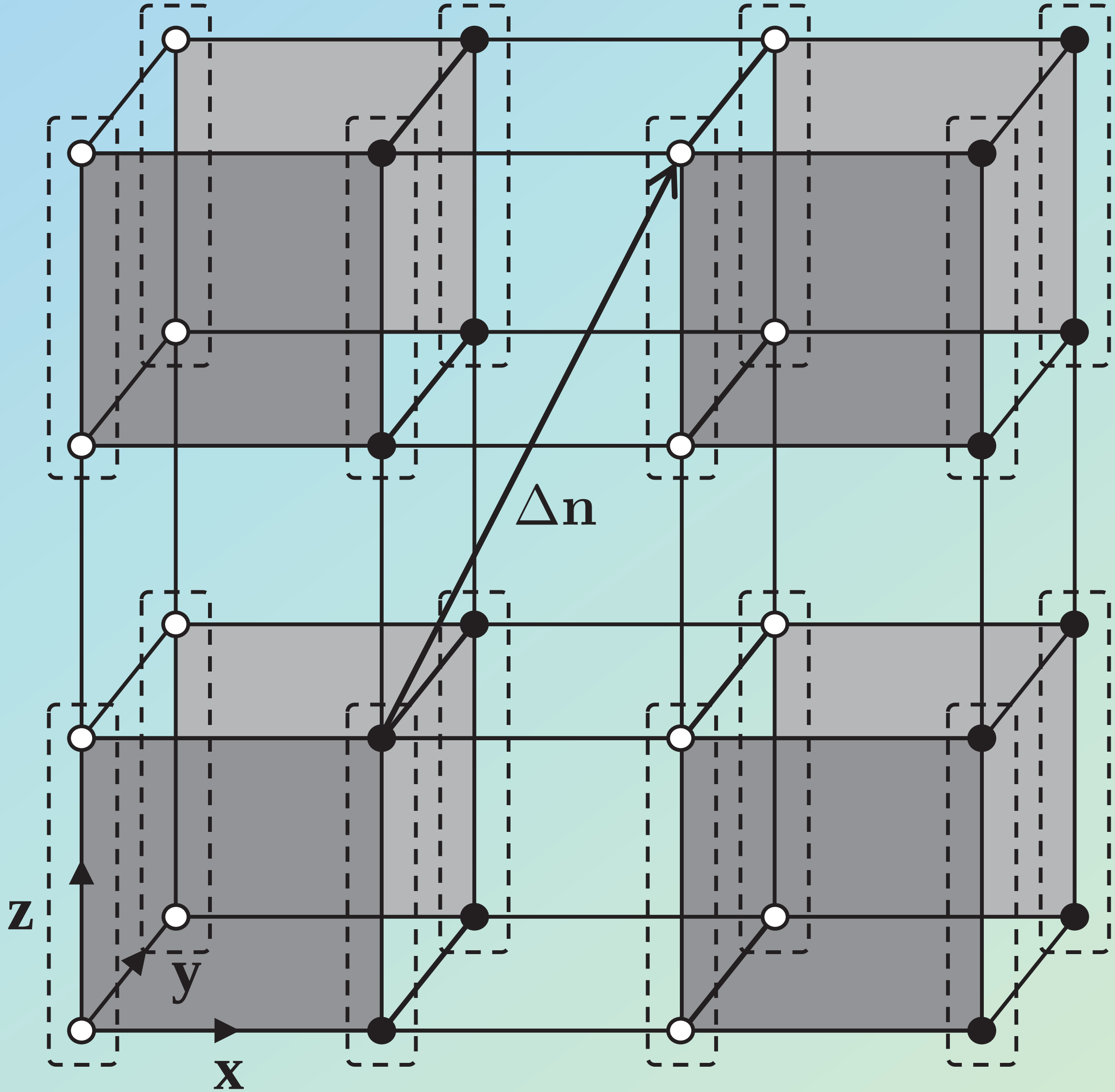
Majies: 3 Dim

Cubic lattice:

Only

$$|\Delta \mathbf{n}| \leq \sqrt{5}$$

$$H = \pm i \sum_{\langle i,j \rangle} \gamma_i \gamma_j$$

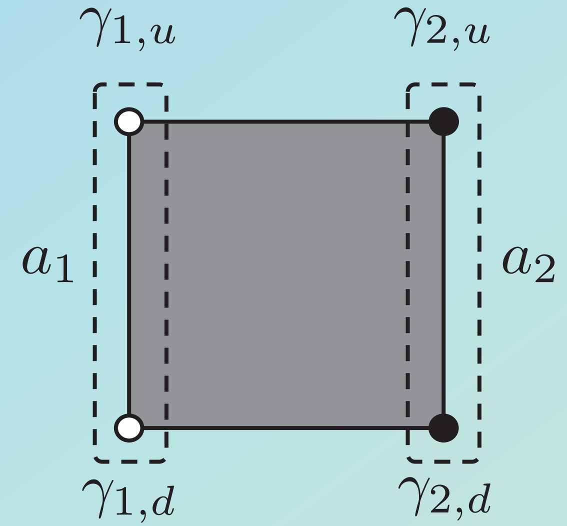


3D Model: superconductor

Fermions:

$$k = 1, 2 \qquad a_k = \frac{\gamma_{k,d} + i\gamma_{k,u}}{2}$$

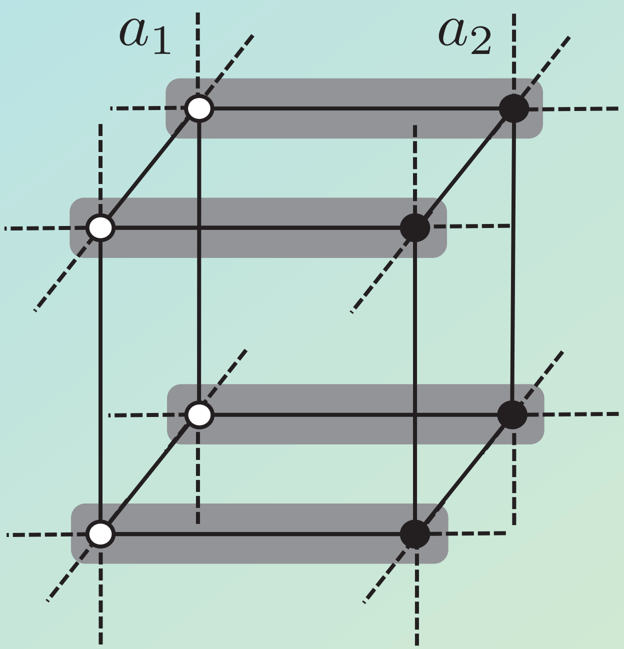
$$a_k^\dagger = \frac{\gamma_{k,d} - i\gamma_{k,u}}{2}$$



Superconducting Ham.

$$H = \sum_{\mathbf{p}} \psi_{\mathbf{p}}^\dagger h(\mathbf{p}) \psi_{\mathbf{p}},$$

$$\psi_{\mathbf{p}} = (a_{1,\mathbf{p}}, a_{1,-\mathbf{p}}^\dagger, a_{2,\mathbf{p}}, a_{2,-\mathbf{p}}^\dagger)^T$$



Topological winding number:

$$T^3 \to S^3$$

3D Model: symmetries

- Impose **time-reversal** symmetry:

$$C_{\mathrm{TR}}^{\dagger} h^{*}(-\mathbf{p}) C_{\mathrm{TR}} = h(\mathbf{p}) \quad C_{\mathrm{TR}} = \sigma^{y} \otimes \mathbb{1}$$

- Impose **particle-hole** symmetry:

$$C_{\mathrm{PH}}^{\dagger} h^{*}(-\mathbf{p}) C_{\mathrm{PH}} = -h(\mathbf{p}) \quad C_{\mathrm{PH}} = \mathbb{1} \otimes \sigma^{x}$$

Topological superconductor of type **DIII**

Can bring Ham in the form: $h(\mathbf{p}) = \begin{pmatrix} 0 & D(\mathbf{p}) \\ D^{\dagger}(\mathbf{p}) & 0 \end{pmatrix}$

with spectrum:

$$E(\mathbf{p}) = \pm \sqrt{\frac{\mathrm{tr}(DD^{\dagger})}{2} \pm \sqrt{\frac{\mathrm{tr}(DD^{\dagger})^{2}}{2} - \mathrm{Det}(DD^{\dagger})}}$$

Eigenstates $|\phi_l(\mathbf{p})\rangle$ eigenvalues $E_l(\mathbf{p})$ for $l = 1, \ldots, 4$

3D Model: winding number

Determine topological order.

Define three-dimensional version of **Chern number**.

Flatten bands:

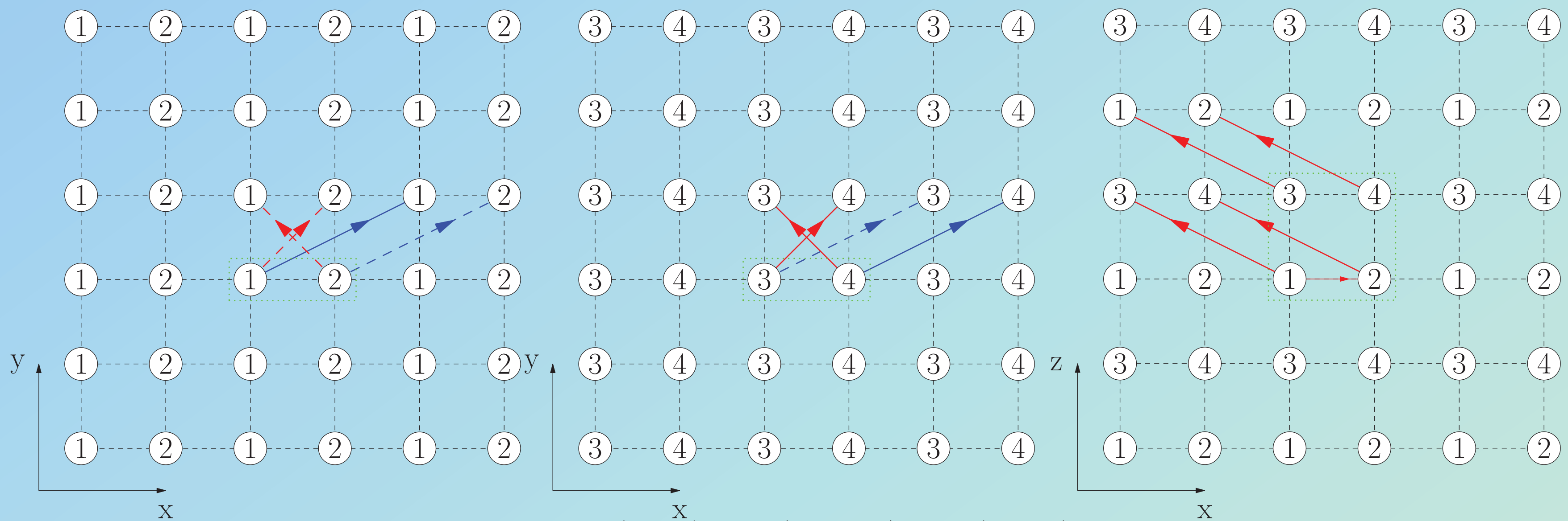
$$Q(\mathbf{p}) = 2\sum_{l=1,2} |\phi_l(\mathbf{p})\rangle \langle \phi_l(\mathbf{p})| - 1\!\!1 \otimes 1\!\!1 = \begin{pmatrix} 0 & q(\mathbf{p}) \\ q^\dagger(\mathbf{p}) & 0 \end{pmatrix}$$

Mapping $T^3 \to S^3$

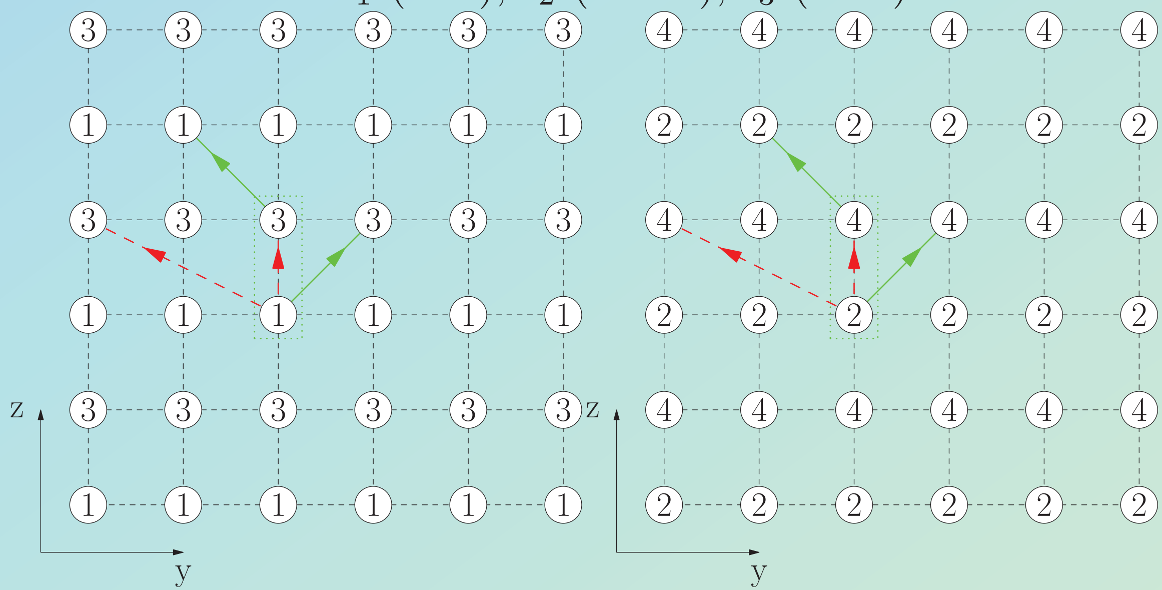
$$\nu = \frac{1}{24\pi^2}\int_{\mathrm{BZ}} d^3p\, \epsilon^{abc}\mathrm{tr}[(q^{-1}\partial_a q)(q^{-1}\partial_b q)(q^{-1}\partial_c q)]$$

Well defined if $E_2(\mathbf{p}) \neq 0$

3D Model: Example

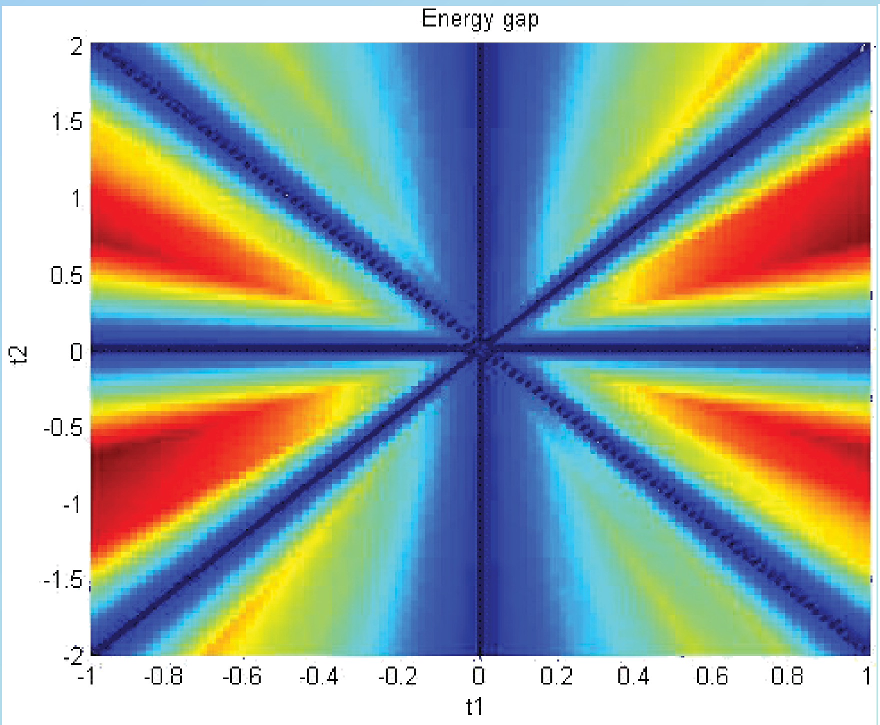


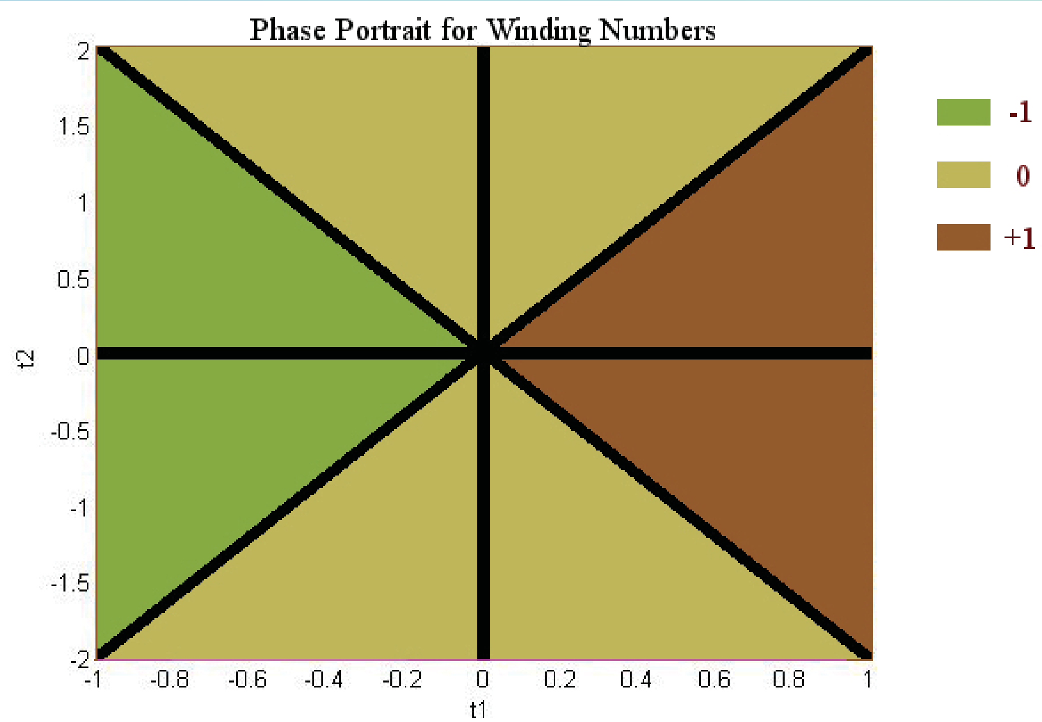
t_1 (Red), t_2 (Green), t_3 (Blue).



3D Model: Example

Energy gap and winding number

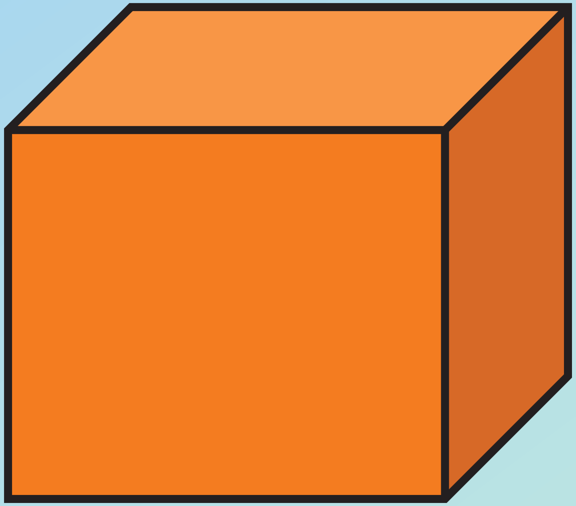


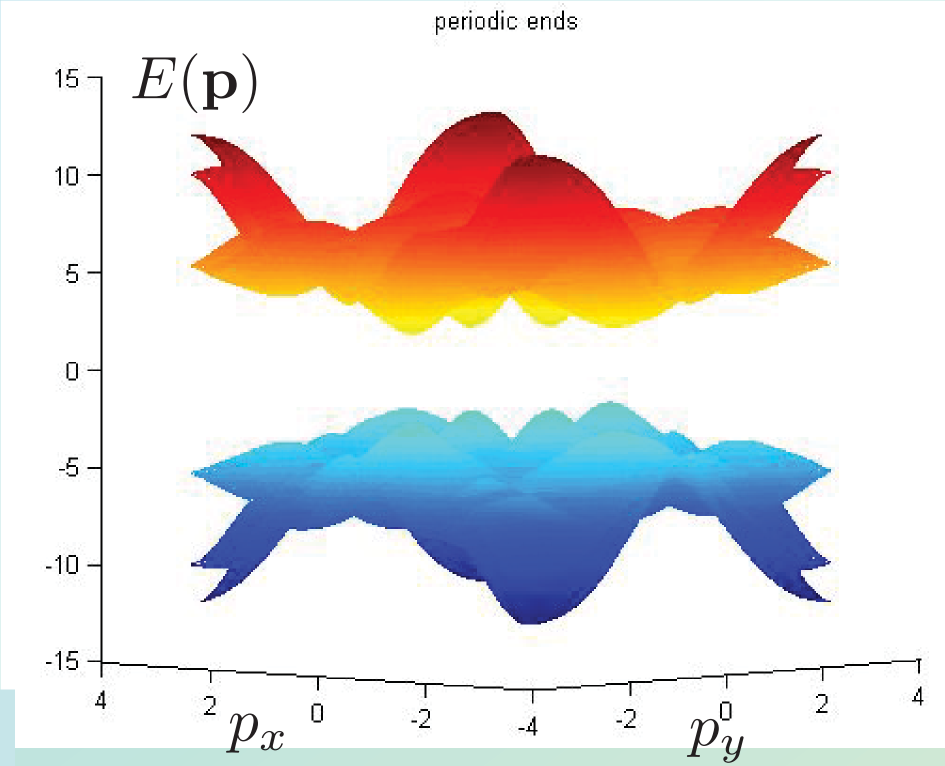


3D Model: Example

Periodic BC in all 3 directions

$\nu = 1$

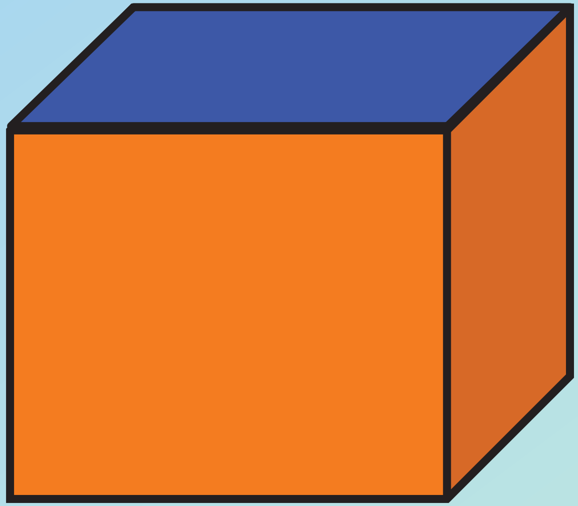


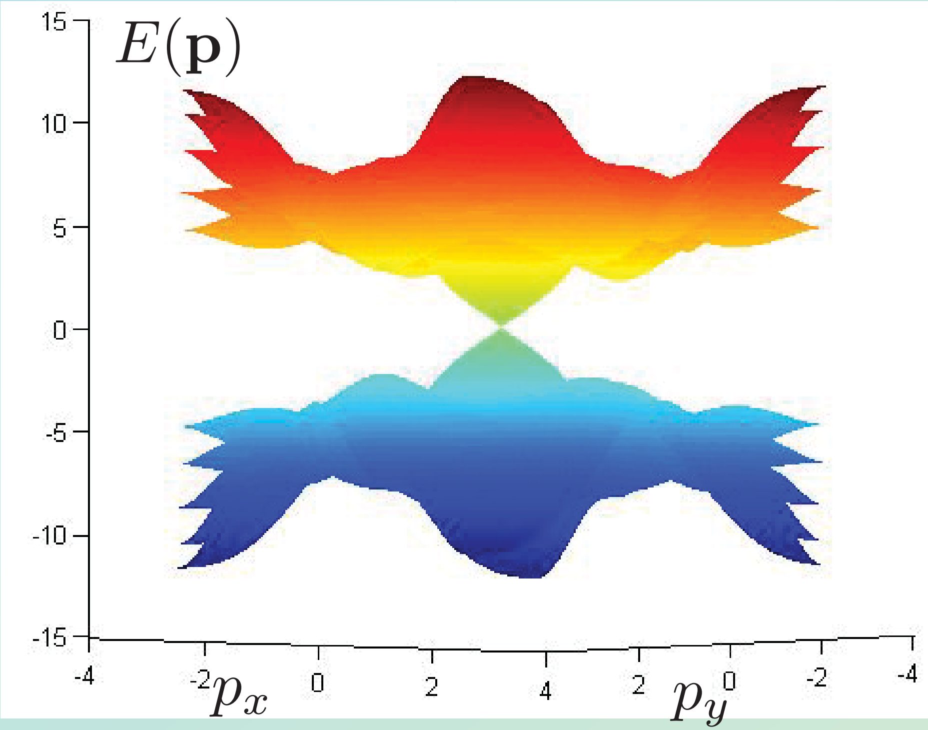


3D Model: Example

Periodic BC in only 2 directions

$\nu = 1$





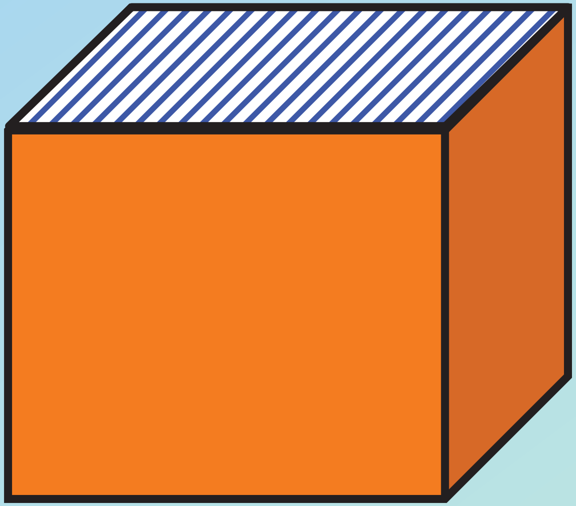
Edge states emerge that manifest themselves as Dirac cones.

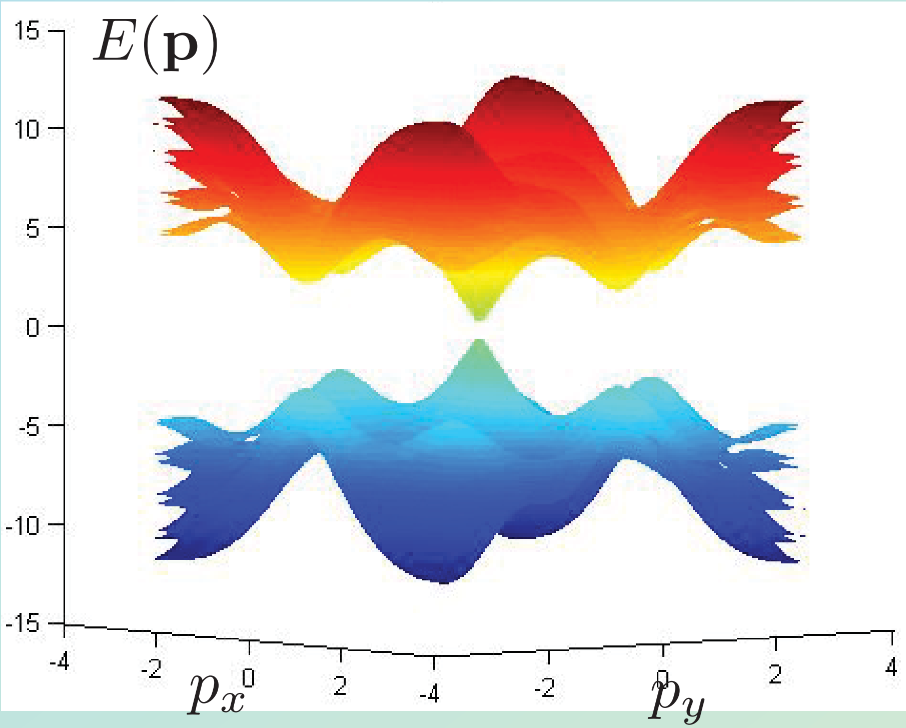
$$\nu = n^L_{\mathrm{Edge}} - n^R_{\mathrm{Edge}}$$

3D Model: Example

Periodic BC in only 2 directions and
"Zeeman term" in y-direction

$\nu = 1$



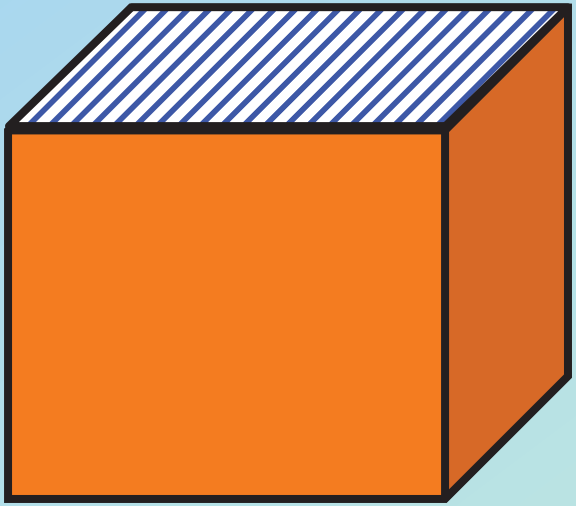


Edge states are gapped.
Does the surface support Majorana fermions?

3D Model: Example

Chern number of bulk = Chern number of surface state

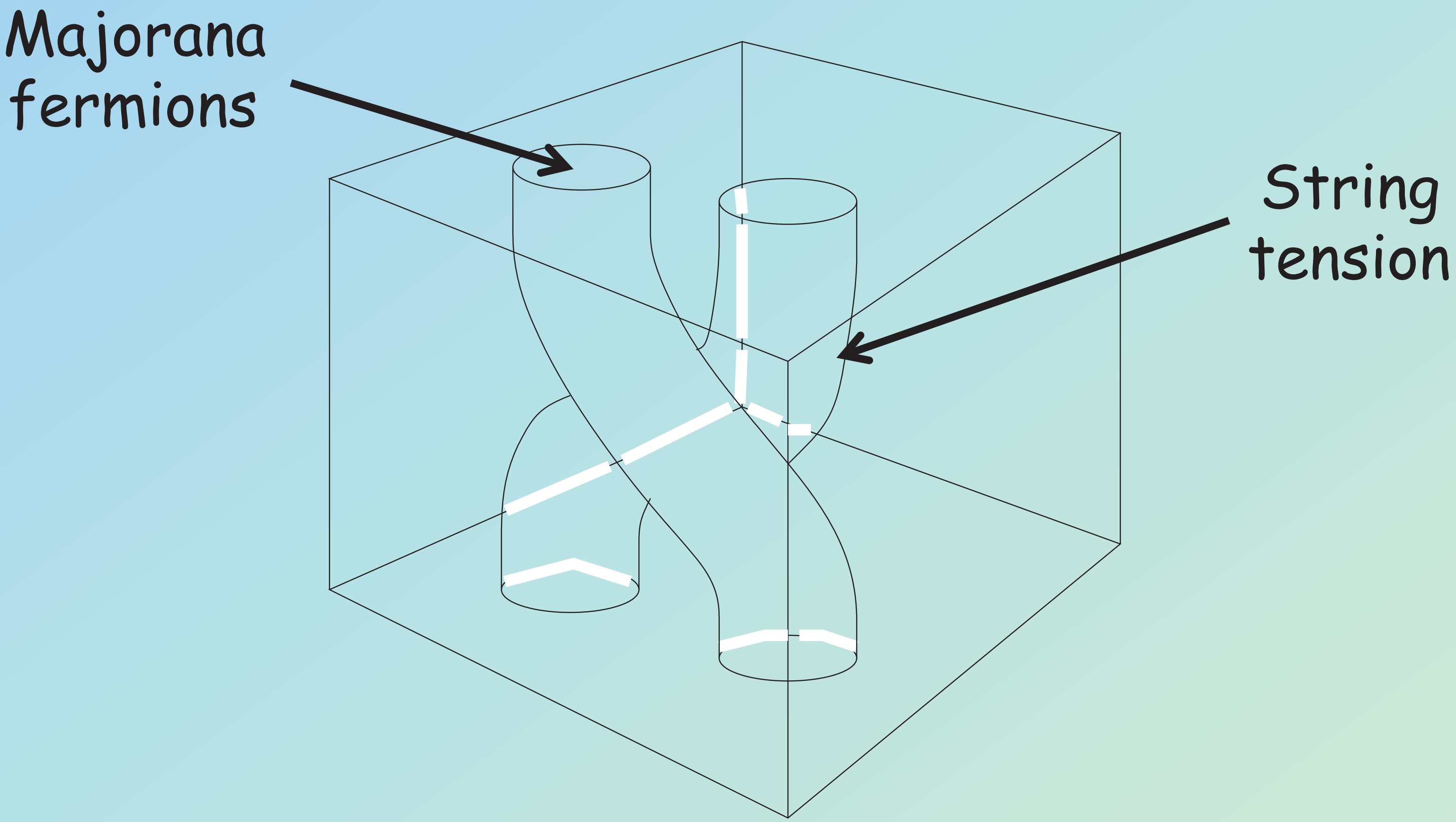
$$\nu_{3D} = \nu_{2D}$$



Majorana fermions can be bound at vortices of the boundary.

3D Model: Example

Vortices have Majies at their end points



Information protected by string tension (similar to cl info)

Conclusions

3D TI & TS provide a laboratory for probing new properties of matter.

- Lab for generating **stable** Majorana fermions:
    - at surface
    - at monopoles in the bulk?
    - Stability against finite temperature

New physics and **new technological applications**