Why should anyone care about computing with anyons?

Jiannis K. Pachos

Toric Code
Non-Abelian
Topological Entropy
Errors
Outlook

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF LEEDS

Isfahan, September 2014

Anyons and Quantum Computation

•Error correction needs a **huge overhead**.

•Instead of performing active error correction let physics do the job.

•Perform QC in a physical medium that is **gapped** and **highly correlated**:

- •**Energy penalty for errors** (gapped).
- •**Make logical errors non-local** (very unlikely).

•Similar to **quantum error correction**, but without active control.

Toric Code: ECC

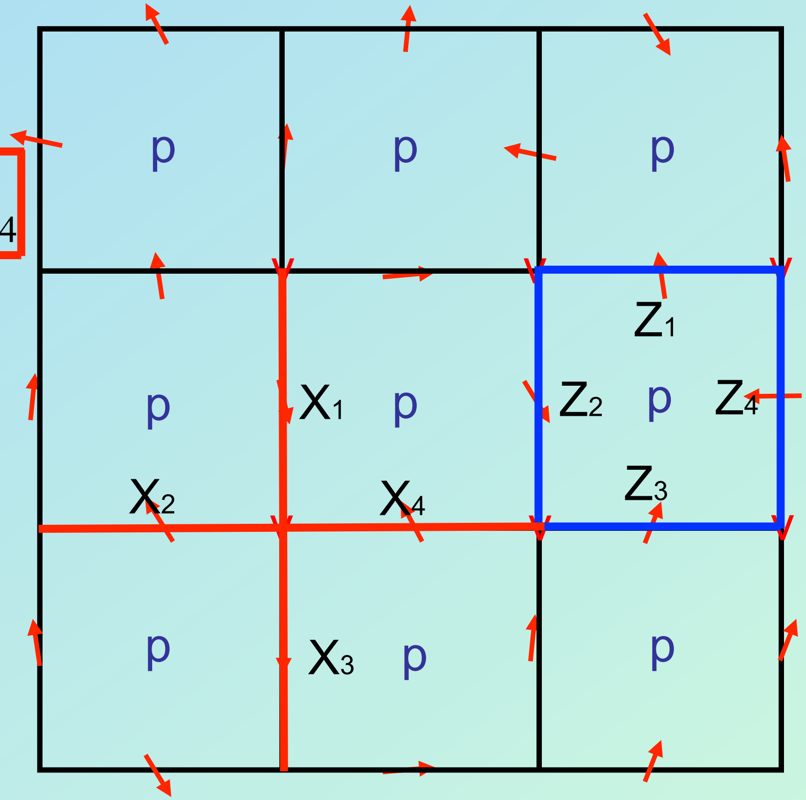
Consider the lattice Hamiltonian

$$H = -\sum_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} - \sum_v X_{v1} X_{v2} X_{v3} X_{v4}$$

Spins on the edges.

Two different types of interactions: ZZZZ or XXXX acting on plaquettes and vertices respectively.

The four spin interactions involve spins of the same vertex/plaquette.



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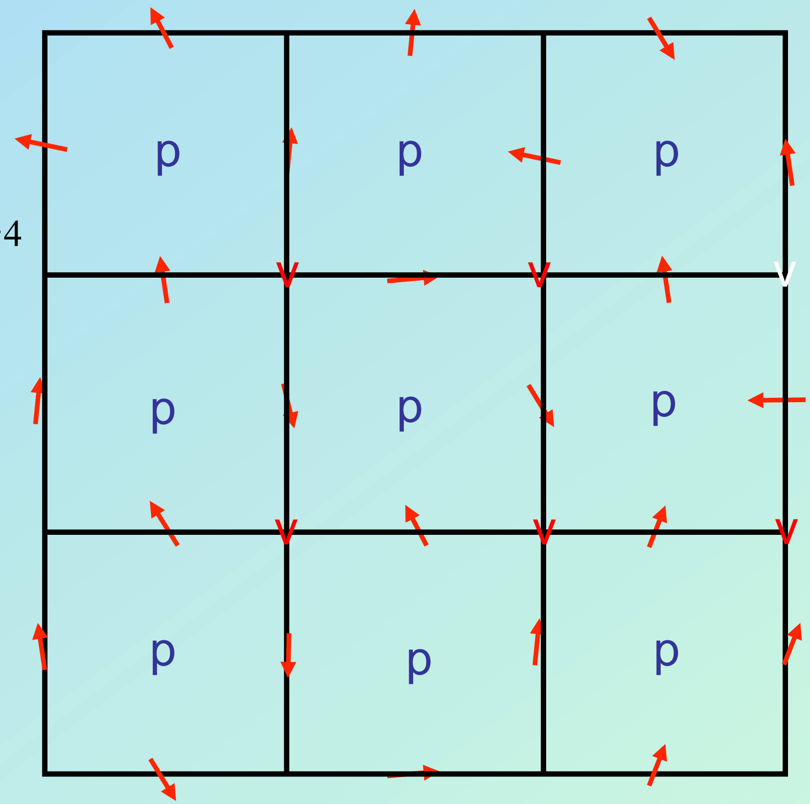

Good **quantum numbers**:

$$[H, Z_{p1}Z_{p2}Z_{p3}Z_{p4}] = 0$$

$$[H, X_{v1}X_{v2}X_{v3}X_{v4}] = 0$$

$$(X_{v1}X_{v2}X_{v3}X_{v4})^2 = 1$$

$$(Z_{p1}Z_{p2}Z_{p3}Z_{p4})^2 = 1$$

⇒eigenvalues of XXXX and ZZZZ: ±1

Also Hamiltonian **exactly solvable**:

$$[X_{v1}X_{v2}X_{v3}X_{v4}, Z_{p1}Z_{p2}Z_{p3}Z_{p4}] = 0$$

Toric Code: ECC

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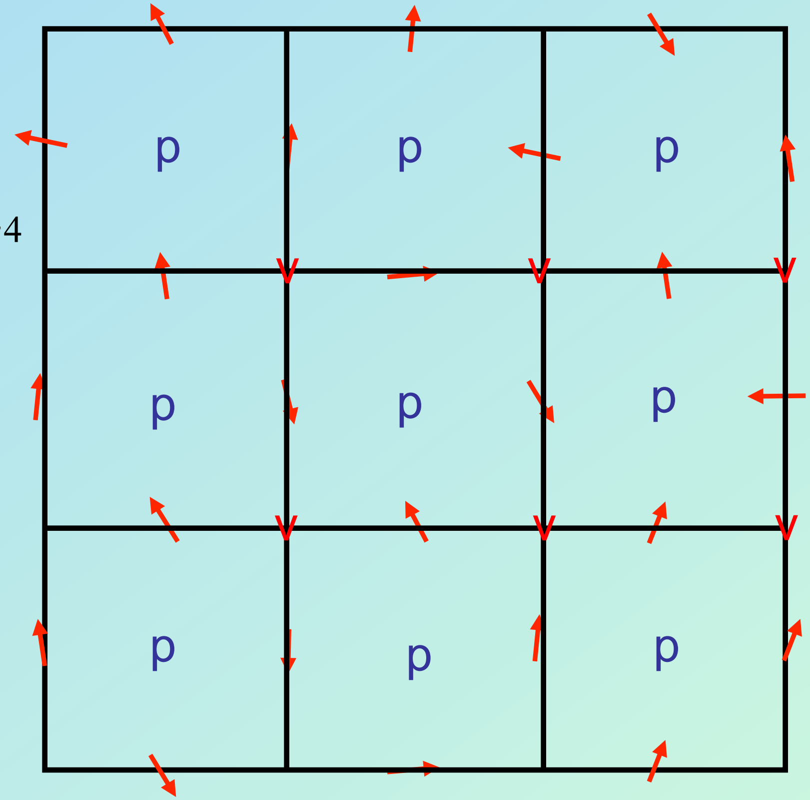

Indeed, the ground state is:

$$|\xi\rangle = \prod_{v} \frac{1}{\sqrt{2}} (\mathrm{I} + X_{v1} X_{v2} X_{v3} X_{v4}) |00...0\rangle$$

The |00…0> state is a ground state of the ZZZZ term.

The (I+XXXX) term projects that state to the ground state of the XXXX term.

Toric Code: ECC

Consider the lattice Hamiltonian

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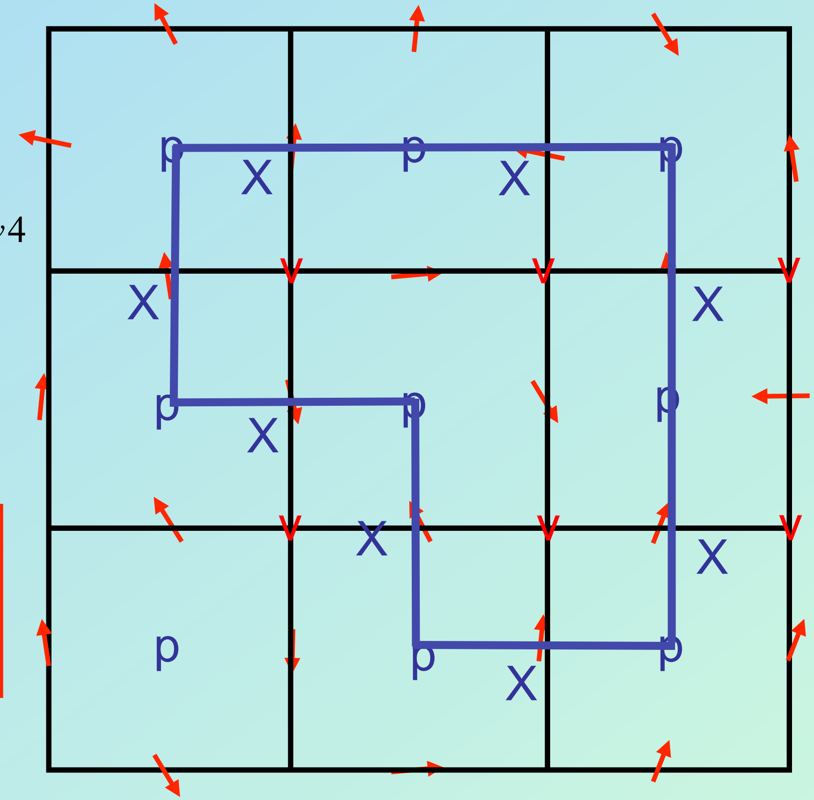


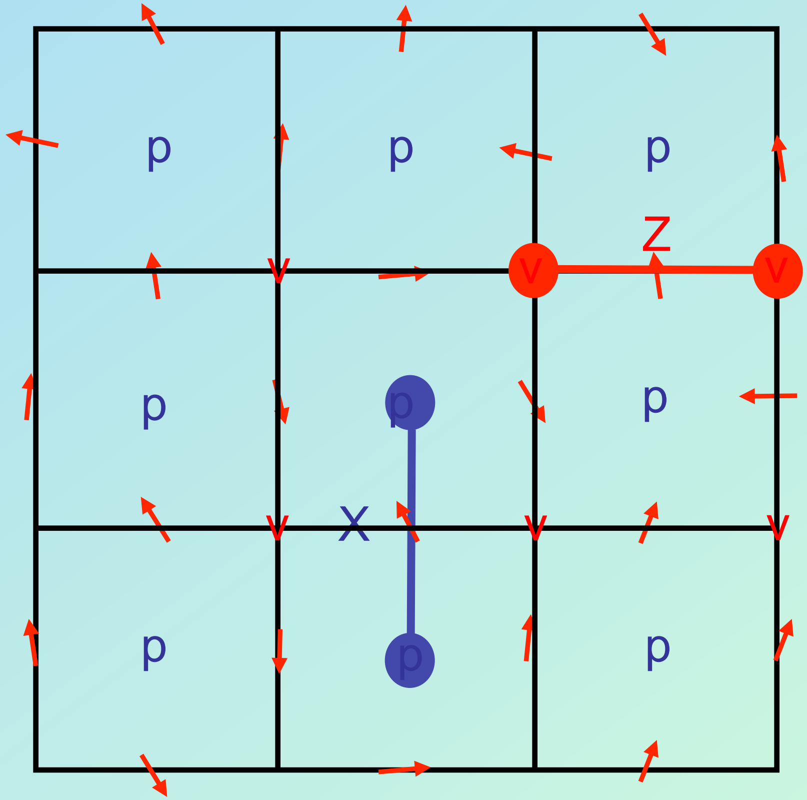
The ground state is a **superposition** of all X **loops**.

It is **stabilized** by the application of **all X loop operators**.

Equivalently for Z loops.

Toric Code: ECC

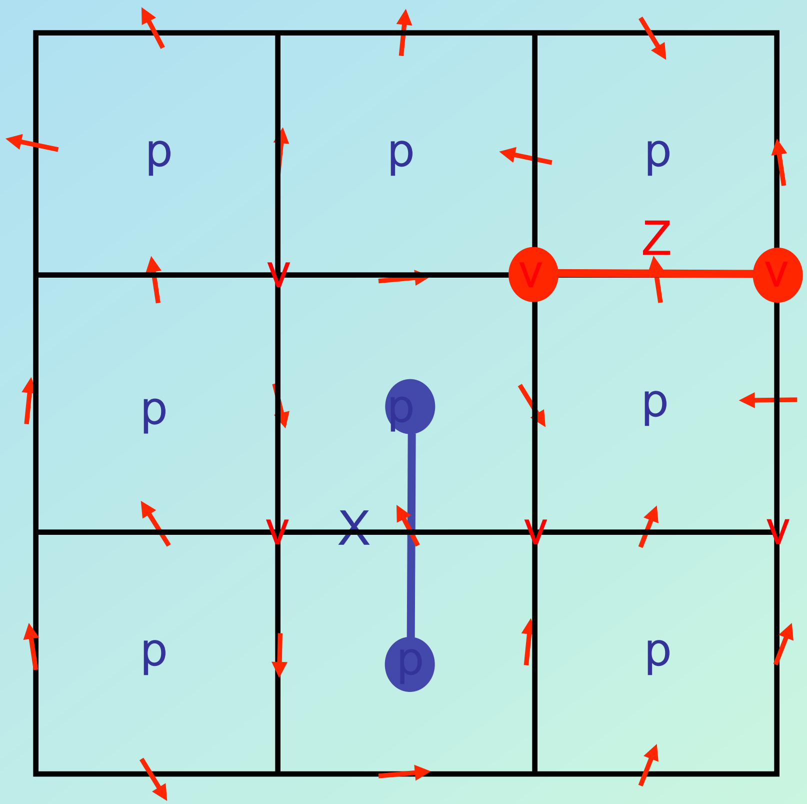
- **Excitations** are produced by Z or X rotations of one spin.
- These rotations **anticommute** with the X- or Z-part of the Hamiltonian, respectively.
- Z excitations on *v* vertices.
- X excitations on *p* plaquettes.



X and Z excitations behave as anyons with respect to each other.

Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.
First create excitations with Z and X rotations.

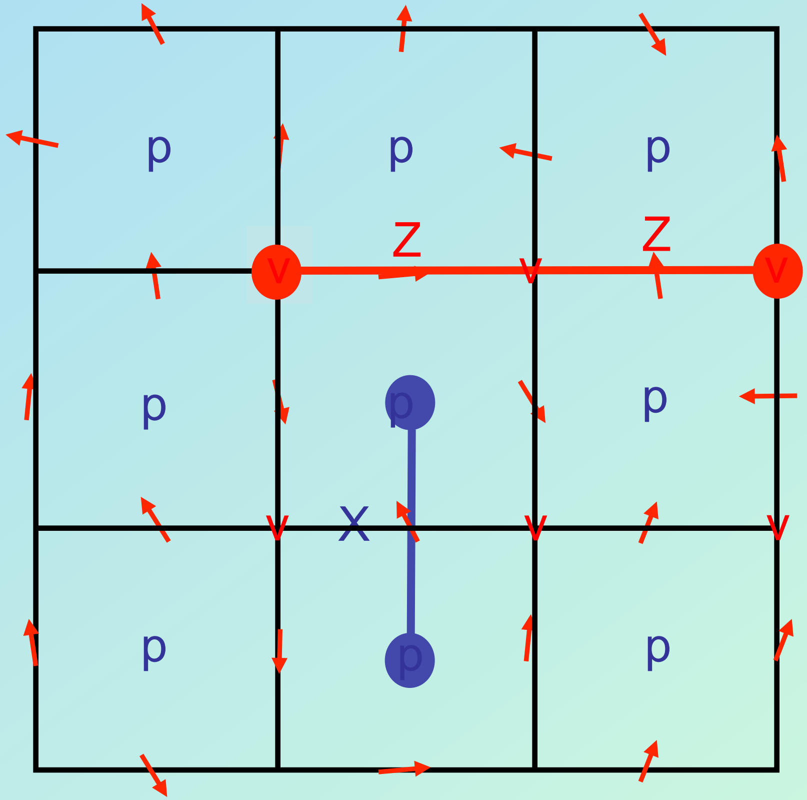


Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.

First create excitations with Z and X rotations.

Then rotate Z excitation around the X one.



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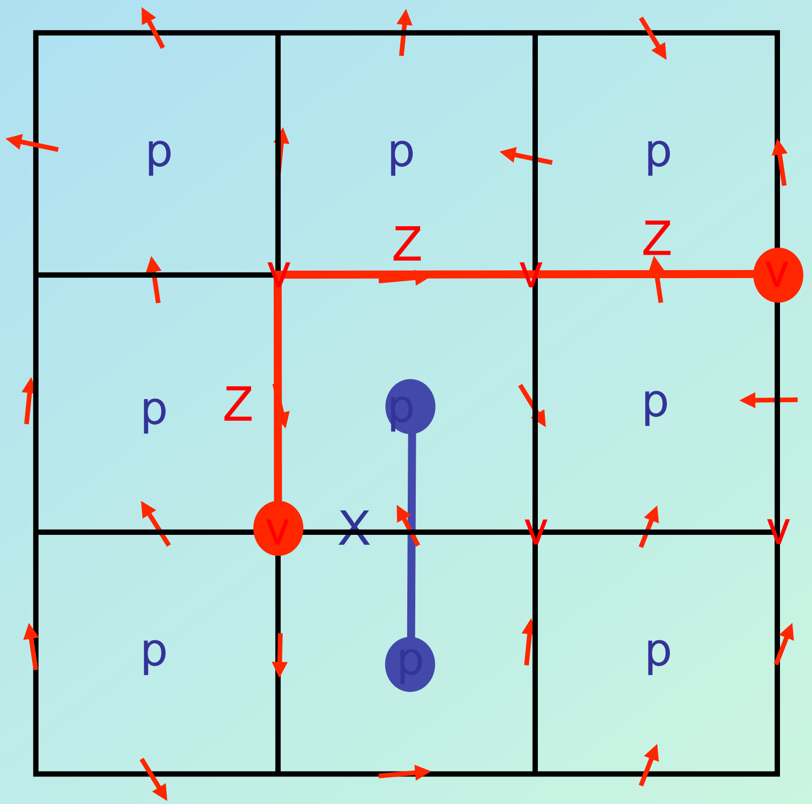

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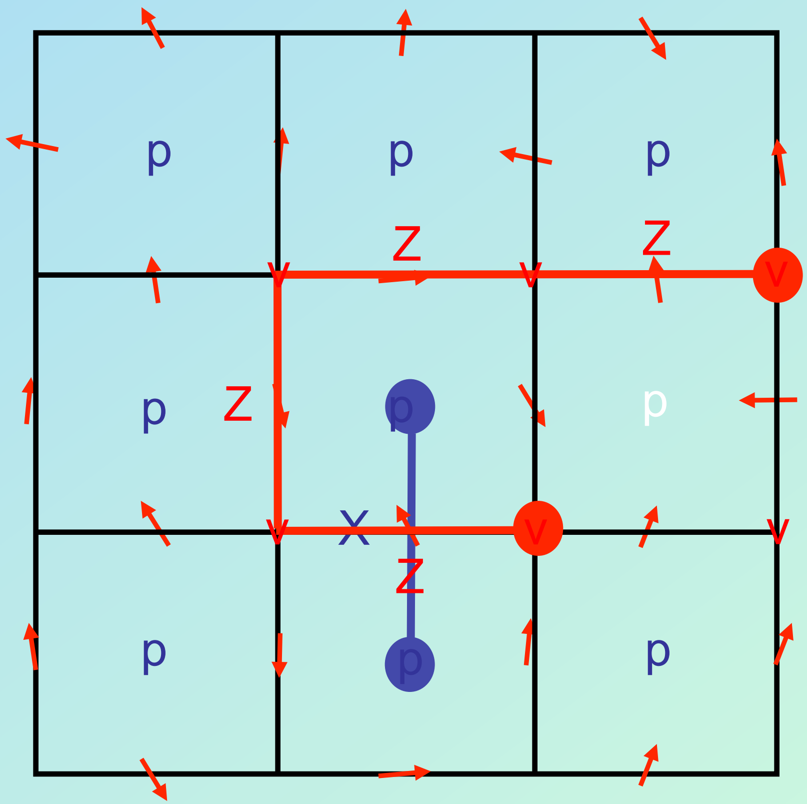

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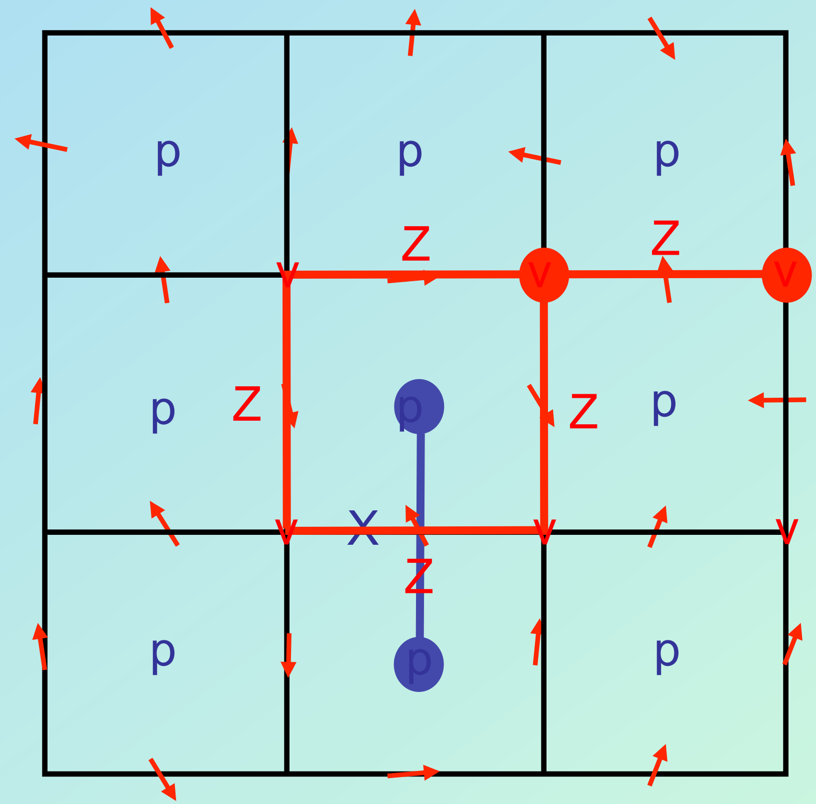


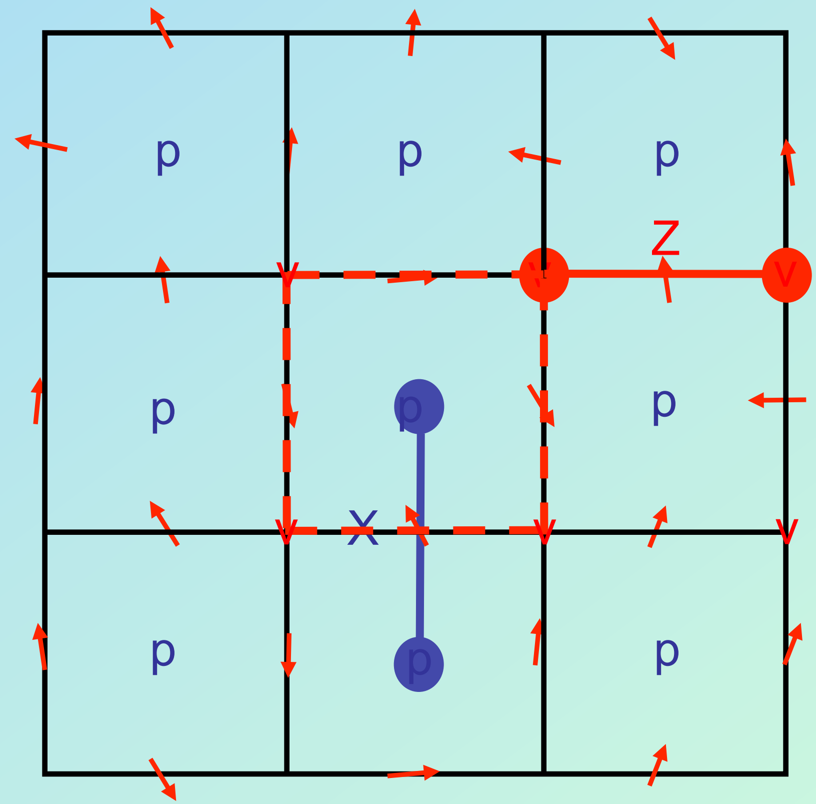
Toric Code: ECC

One can demonstrate the anyonic statistics between X and Z.

First create excitations with Z and X rotations.

Then rotate Z excitation around the X one.

This results in plaquette operator detecting the X excitation. Gives -1



$$|Final\rangle = Z_4 Z_3 Z_2 Z_1 |X\rangle = (Z_4 Z_3 Z_2 Z_1) X_3 |\xi\rangle$$
$$= -X_3 (Z_4 Z_3 Z_2 Z_1) |\xi\rangle = -|Initial\rangle$$

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Anyonic statistics

After a complete rotation of an X anyon around a Z anyon (two successive exchanges) the resulting state gets a phase π (a minus sign): hence **ANYONS** with statistical angle $\pi / 2$

A property we used is that $X_4 X_3 X_2 X_1 |\xi\rangle = |\xi\rangle$

Toric Code: ECC

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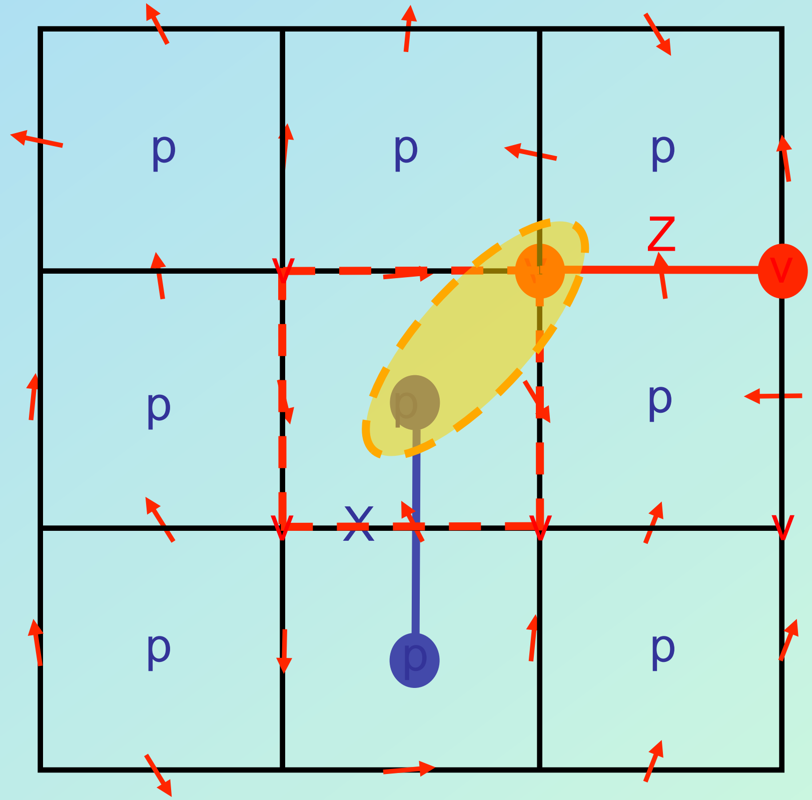

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Toric Code: Anyons

Hence Toric Code has particles:

$$1, e\ (Z), m\ (X), \varepsilon(\text{fermion})$$

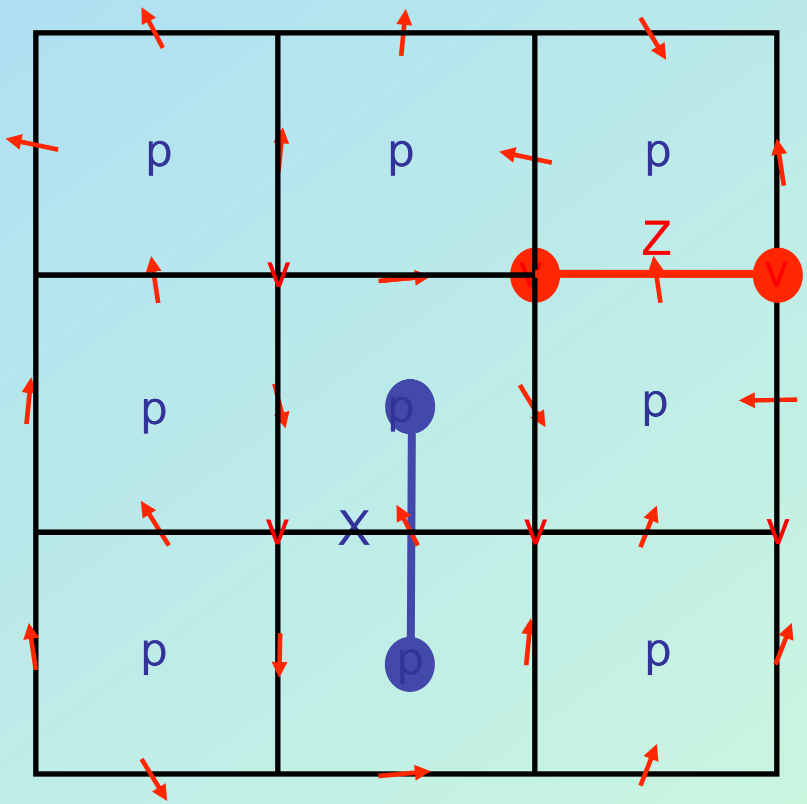
Fusion rules:

$$e \times e = 1, m \times m = 1, \varepsilon \times \varepsilon = 1$$

$$e \times m = \varepsilon, e \times \varepsilon = m, m \times \varepsilon = e$$

Fusion moves: F are trivial

Braiding moves R: $R^{\varepsilon}_{em} = i, R^{1}_{\varepsilon\varepsilon} = -1$



Toric Code: Encoding

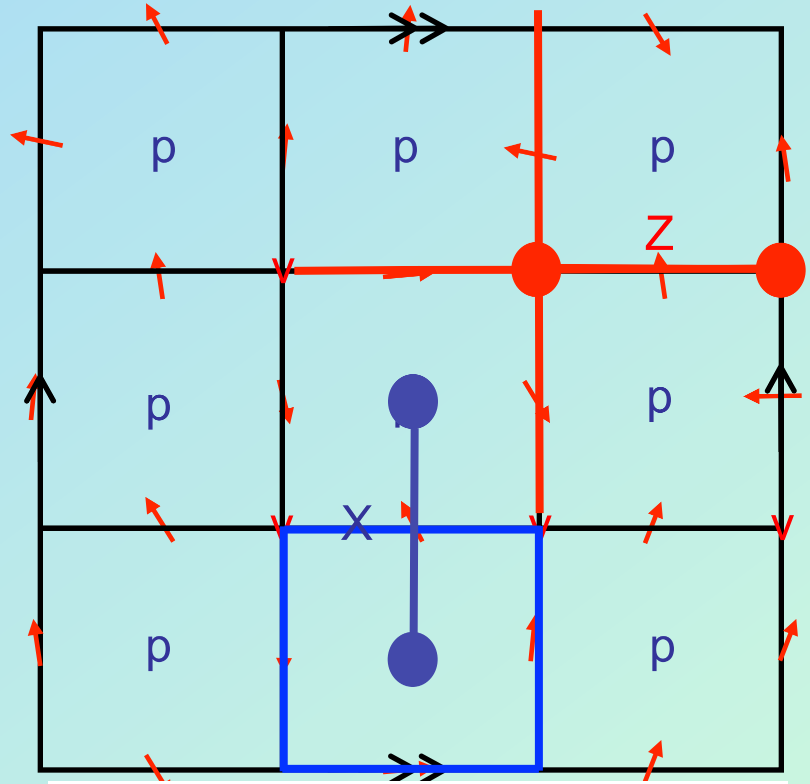
Toric code as a **quantum error correcting code**.

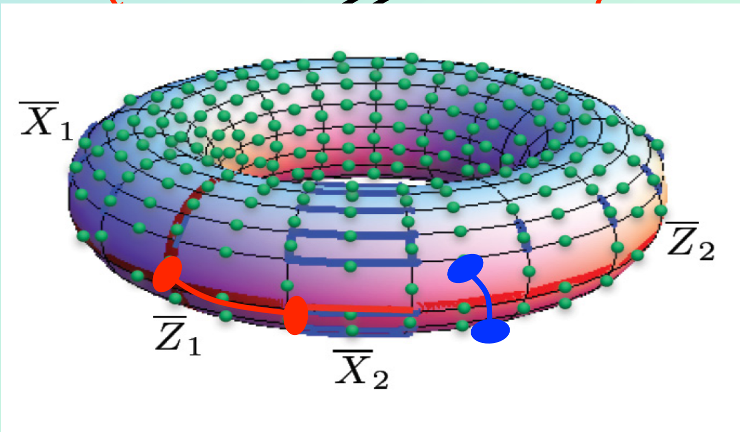
Consider periodic boundary conditions: TORUS of size L

Errors: Anyons

Error correction: detect anyons/errors and connect shortest distance between the same type of anyons.

$$\prod_v X_{v1} X_{v2} X_{v3} X_{v4} = \prod_p Z_{p1} Z_{p2} Z_{p3} Z_{p4} = 1$$





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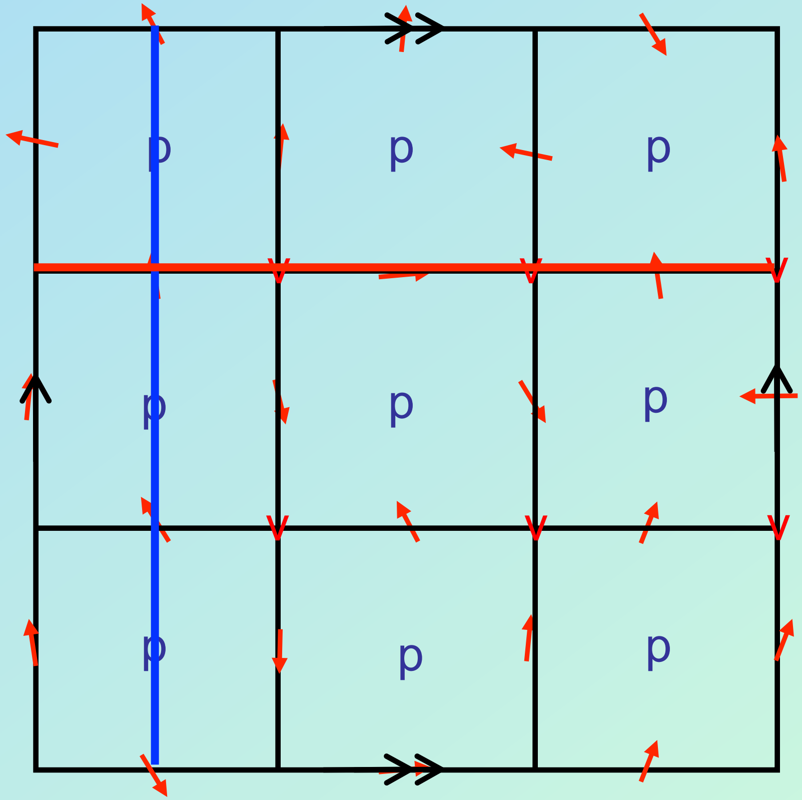

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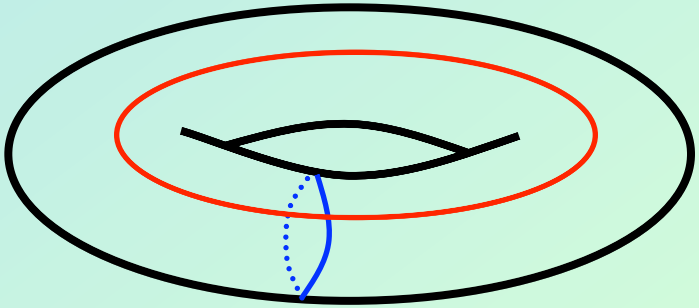


Logical Gates: non-trivial loops

Toric Code: Encoding

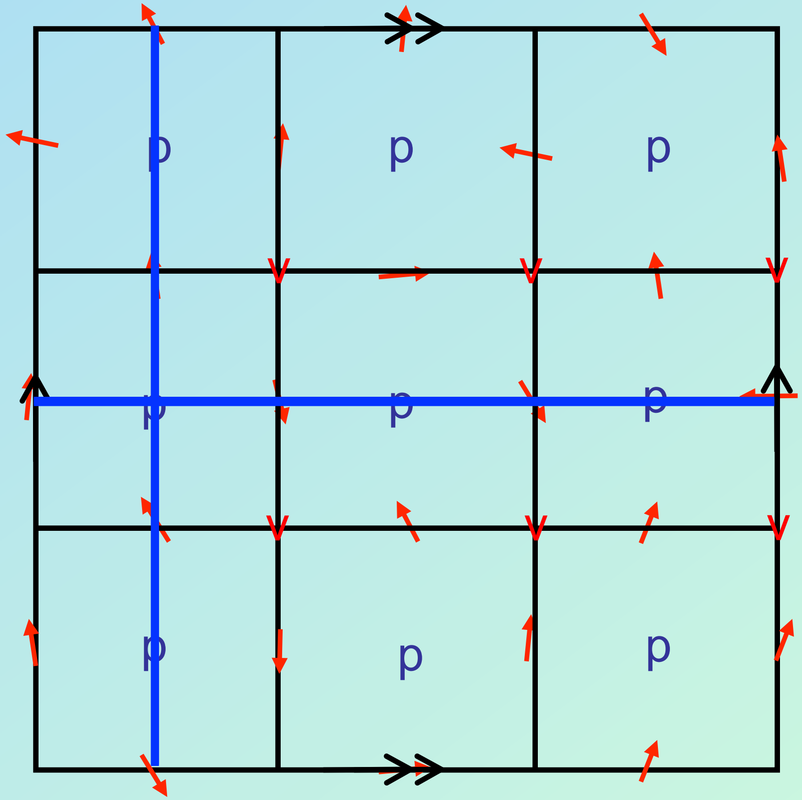
Logical Space and Gates

$$|\Psi_1\rangle$$

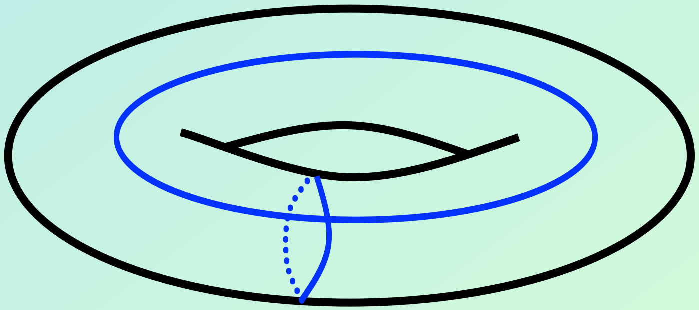
$$|\Psi_2\rangle = C_X^1 |\Psi_1\rangle$$

$$|\Psi_3\rangle = C_X^2 |\Psi_1\rangle$$

$$|\Psi_4\rangle = C_X^2 C_X^1 |\Psi_1\rangle$$



Can store two qubits and perform Clifford group operations!



Higher genus, g, stores 2g qubits.

Quantum Double Models

Toric Code is an example of **quantum double models**.

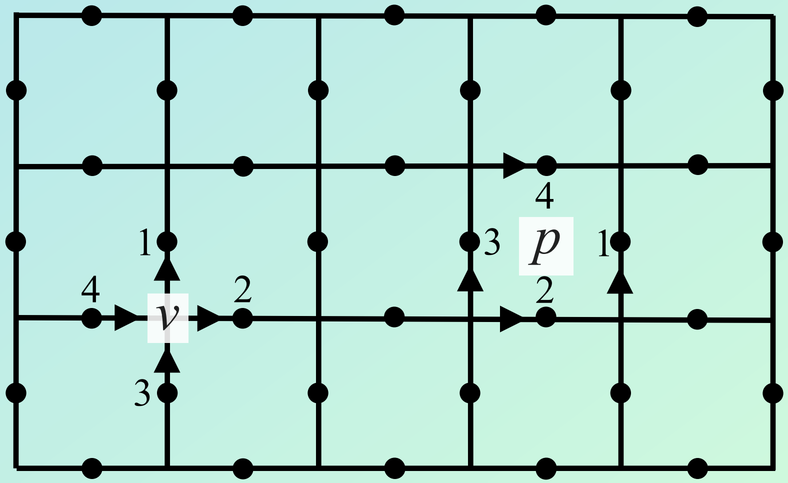
Corresponding group Z_2 ={1,e;e^2=1} that gives rise to qubit states |1>,|e>.

Imagine a general **finite group** G={g_1, g_2, ...,g_d} and the corresponding qudit with states |g_i>, i=1,...,d.

Consider a **qudit** positioned at each **edge** of a square lattice.

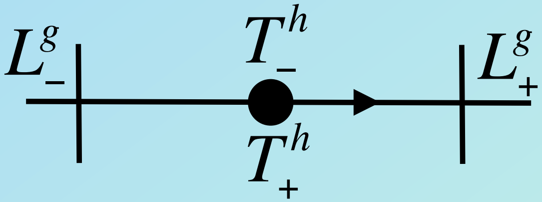
Define **orientation** on the lattice:

Upwards and Rightwards



Quantum Double Models

Define operators:



$$L_+^g|z\rangle = |gz\rangle, \quad L_-^g|z\rangle = |zg^{-1}\rangle, \quad T_+^h|z\rangle = \delta_{h,z}|z\rangle, \quad T_-^h|z\rangle = \delta_{h^{-1},z}|z\rangle$$

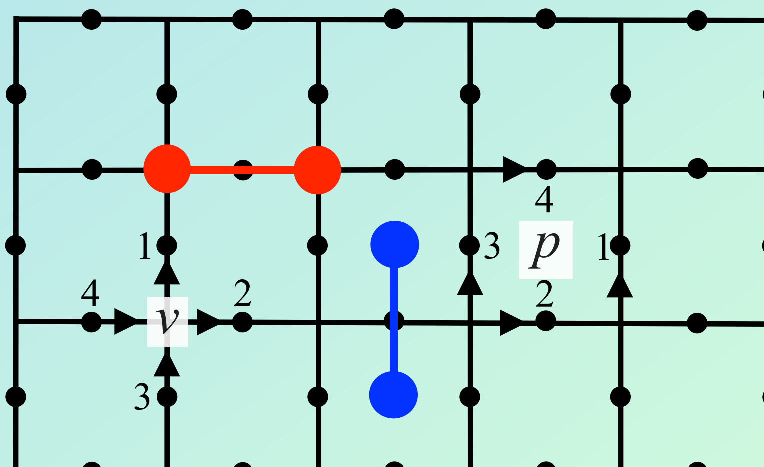
$$A(v) = \frac{1}{|G|}\sum_{g\in G} L_+^g(e_1)L_+^g(e_2)L_-^g(e_3)L_-^g(e_4), \quad B(p) = \sum_{h_1\ldots h_4=1} T_-^{h_1}(e_1)T_-^{h_2}(e_2)T_+^{h_3}(e_3)T_+^{h_4}(e_4)$$

Hamiltonian and ground state:

$$H = -\sum_v A(v) - \sum_p B(p)$$

$$A(v)|\xi\rangle = |\xi\rangle$$

$$B(p)|\xi\rangle = |\xi\rangle$$



Quantum Double Models

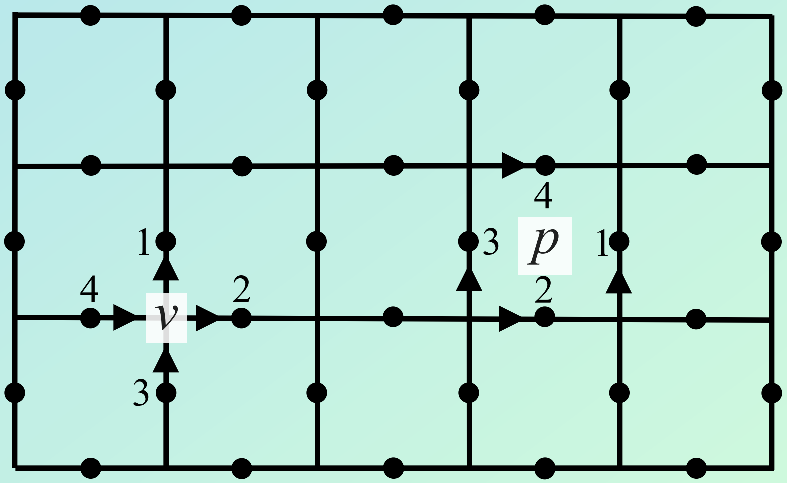
This is also an **error correcting code** defined from the **stabilizer formalism**.

The errors are **anyons**, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

These properties can be explicitly determined.

Examples: $D(Z_2)$, $D(Z_2 \times Z_2)$, $D(S_3)$

$S_3=\{1,x,y,y^2,xy,xy^2;\ x^2=1,y^3=1\}$

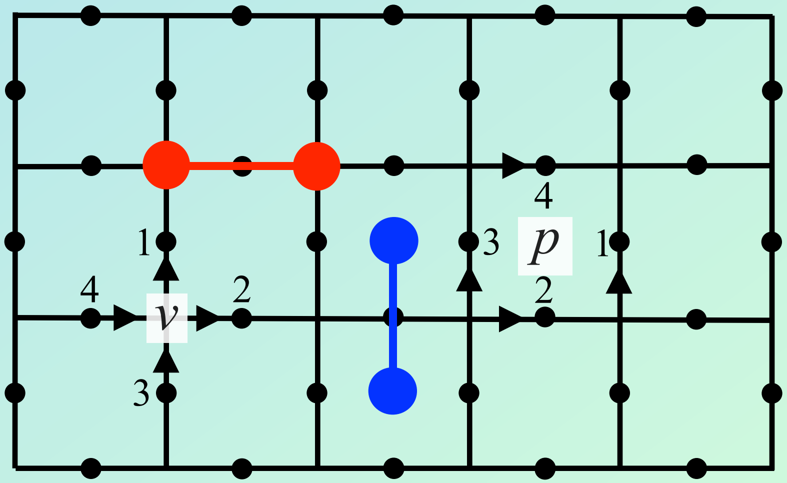


Quantum Double Models

This is also an **error correcting code** defined from the **stabilizer formalism**.

The errors are **anyons**, Abelian or non-Abelian, with the corresponding fusion rules, B and F matrices.

Information can be encoded in the fusion space of non-Abelian anyons and manipulated by braiding them.

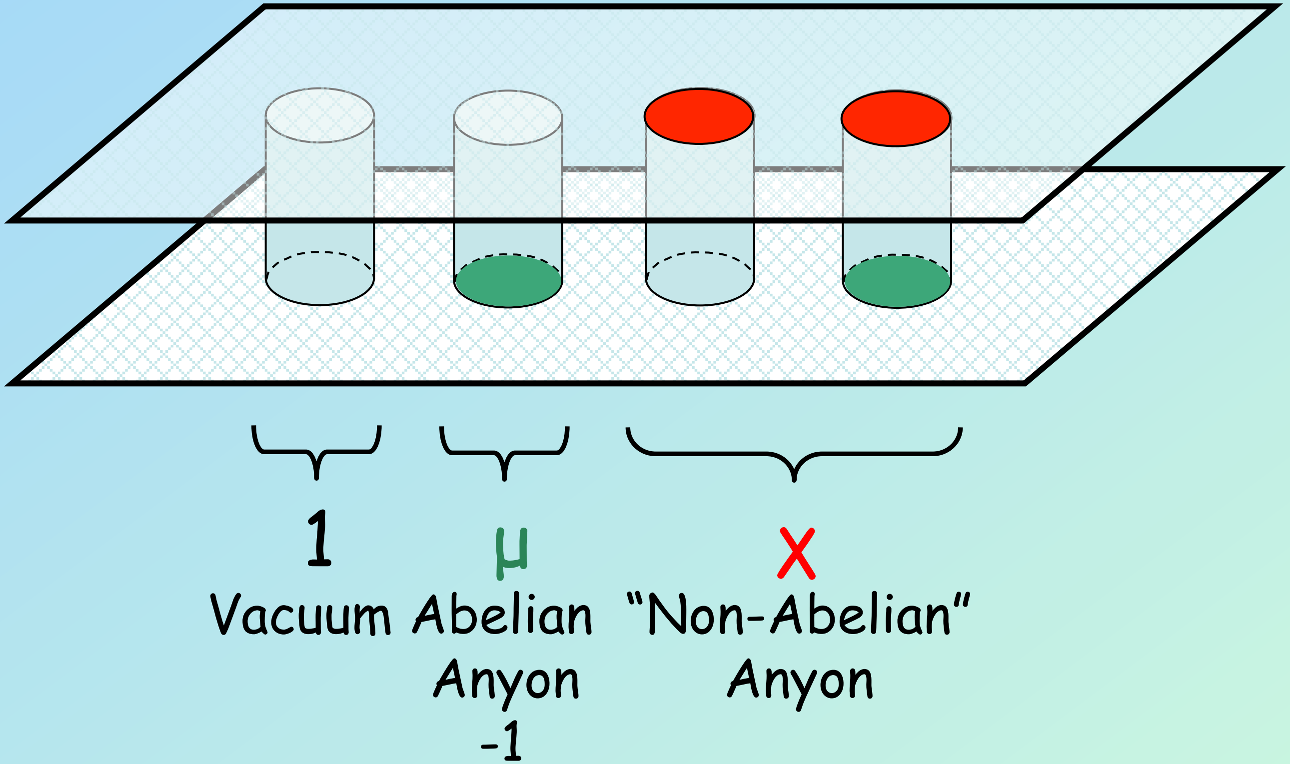


Realizations:

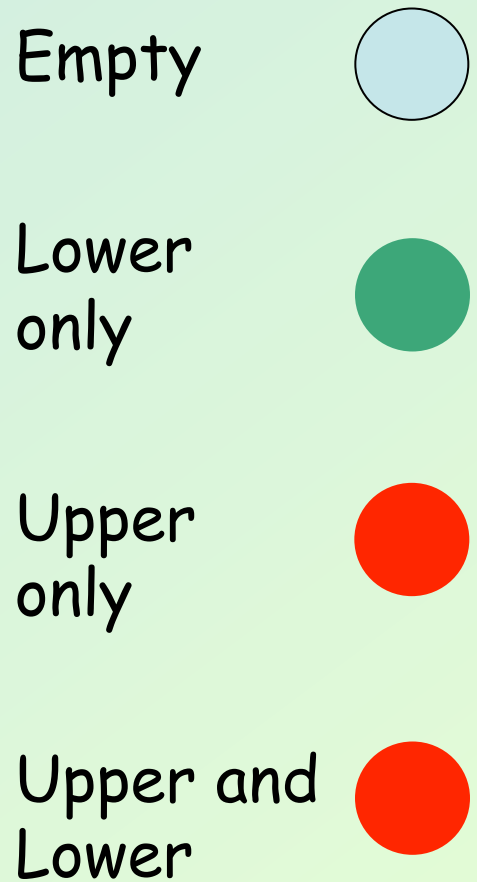
Josephson junctions, photons, optical lattices,...

From Abelion to Nonabelion

- The scheme: $D(Z_2 \times Z_2)$ [or $D(S_3)$] similar to two toric codes



Hamiltonian bird's eye view:

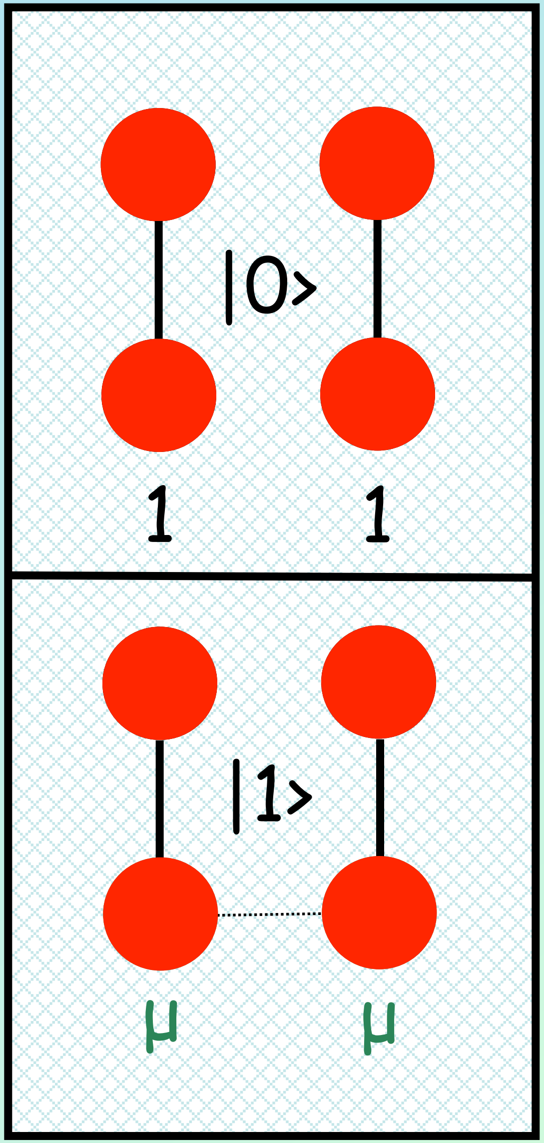


From Abelion to Nonabelion

- Encode information in **fusion channels**:

$$\mu \times \mu = 1, \qquad \chi \times \chi = 1 + \mu$$

- Qubit needs four anyons
- Logical **|0>** when each pair fuses to the **vacuum 1**
- Logical **|1 >** when each pair fuses to **μ**
- 1, μ indistinguishable to local operations when dressed with χ
- **Measurement by fusion**

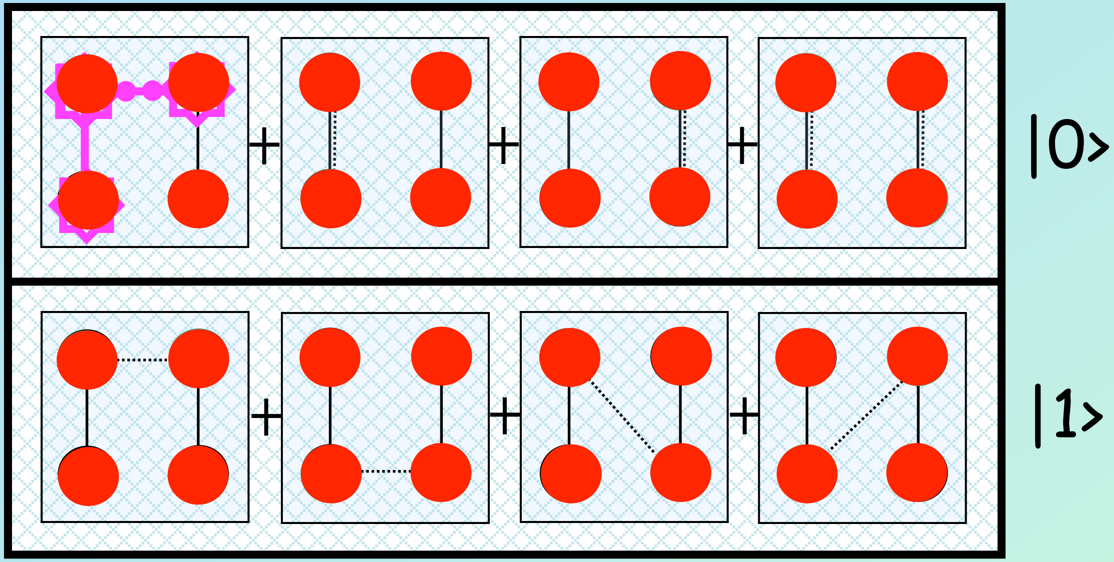


From Abelion to Nonabelion

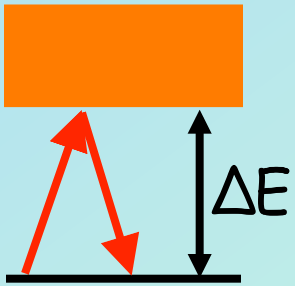
- Fault-tolerance
  - Phase flips ●
  - Bit flips ●

  by **non-local** operators only

  → topo. protection



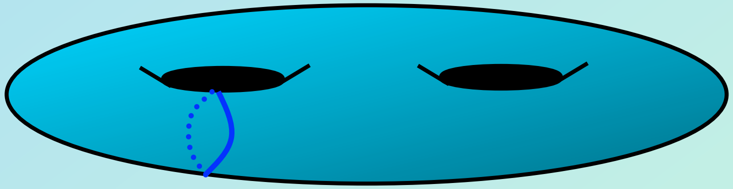
Energy gap present even during gate operations

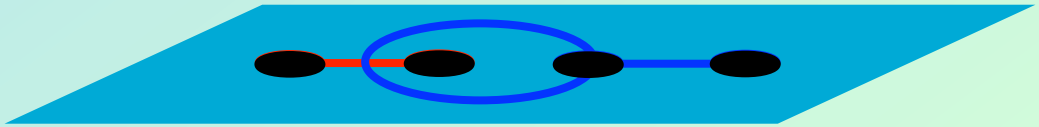


- **Redundancy** and non-locality protects against **virtual transitions**
- Braiding is only Abelian.

Summary

- **Quantum Double models**:
  - Toric Code
  - Abelian encoding and quantum computation
  - Non-Abelian models
- **Degenerate** encoding states
- **Energy gap** above encoding space
- **Manipulations** of code space: higher **genus** surface or with **anyons** or **punctures**: encoding Hilbert space becomes larger.





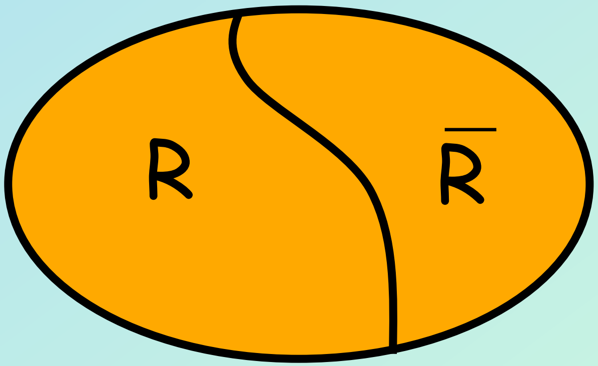
Further

- **Detecting** topological order
    - Topological entropy
- **Errors** and topological order
    - **Topological memories**
    - **Protection against errors**

Topological Entropy

- Pure system $|\xi\rangle$
- Partition in R and $\bar{R}$ with boundary ∂R
- Reduced density matrix of R:

$$\rho_R = \mathrm{tr}_{\bar{R}} |\xi\rangle\langle\xi|$$



- Von Neumann entropy:

$$S_R = tr(\rho_R \ln \rho_R)$$

- We expect:

$$S_R = \alpha |\partial R| + \gamma + \varepsilon(|\partial R|^{-1})$$

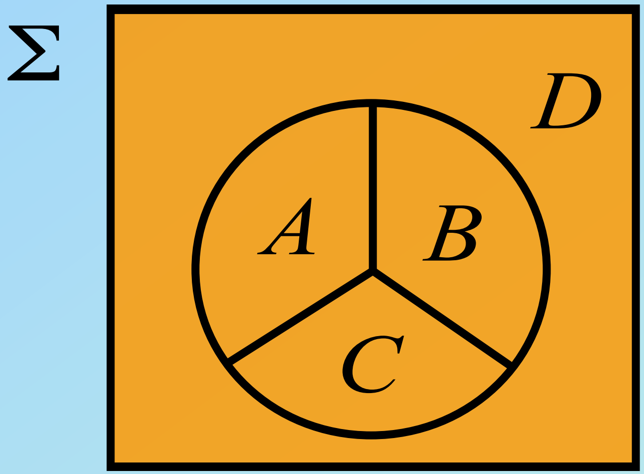
0

- Topological entropy: $\gamma = \ln D, \quad D = \sqrt{\sum_q d_q^2}$

!

Topological Entropy

- Consider partition of single system Σ:



System is **gapped** ->

finite correlation length

Size of areas -> **infinity**

$$\varepsilon(|\partial R|^{-1}) \to 0$$

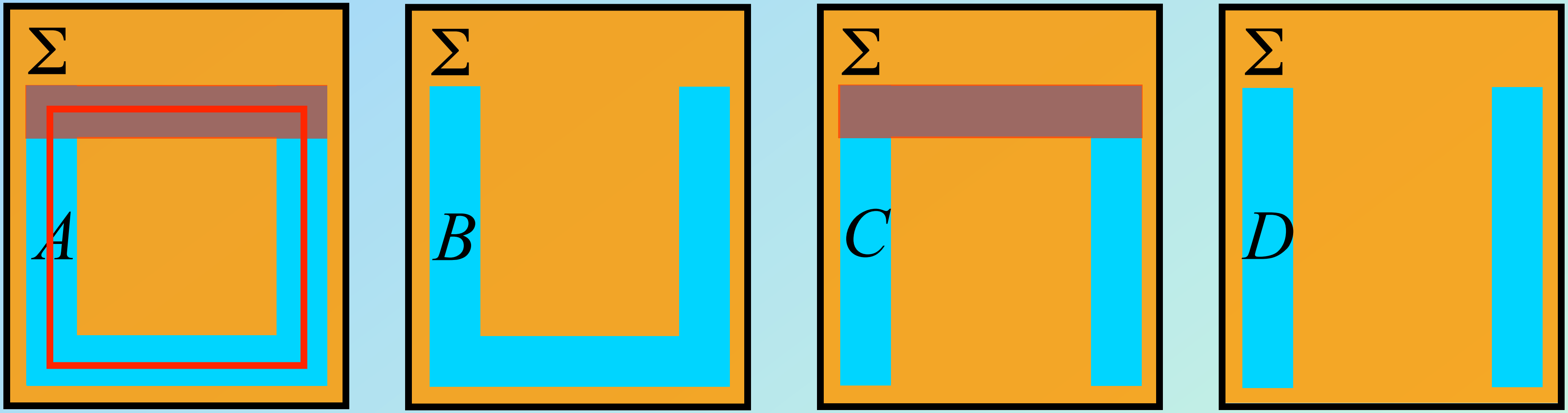
- Topological entropy [Kitaev & Preskill]:

$$\gamma = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + S_{ABC}$$

The area terms disappear! $S_R = \alpha |\partial R| + \gamma$

Topological Entropy

- Consider four different partitions:



S_A S_B S_C S_D

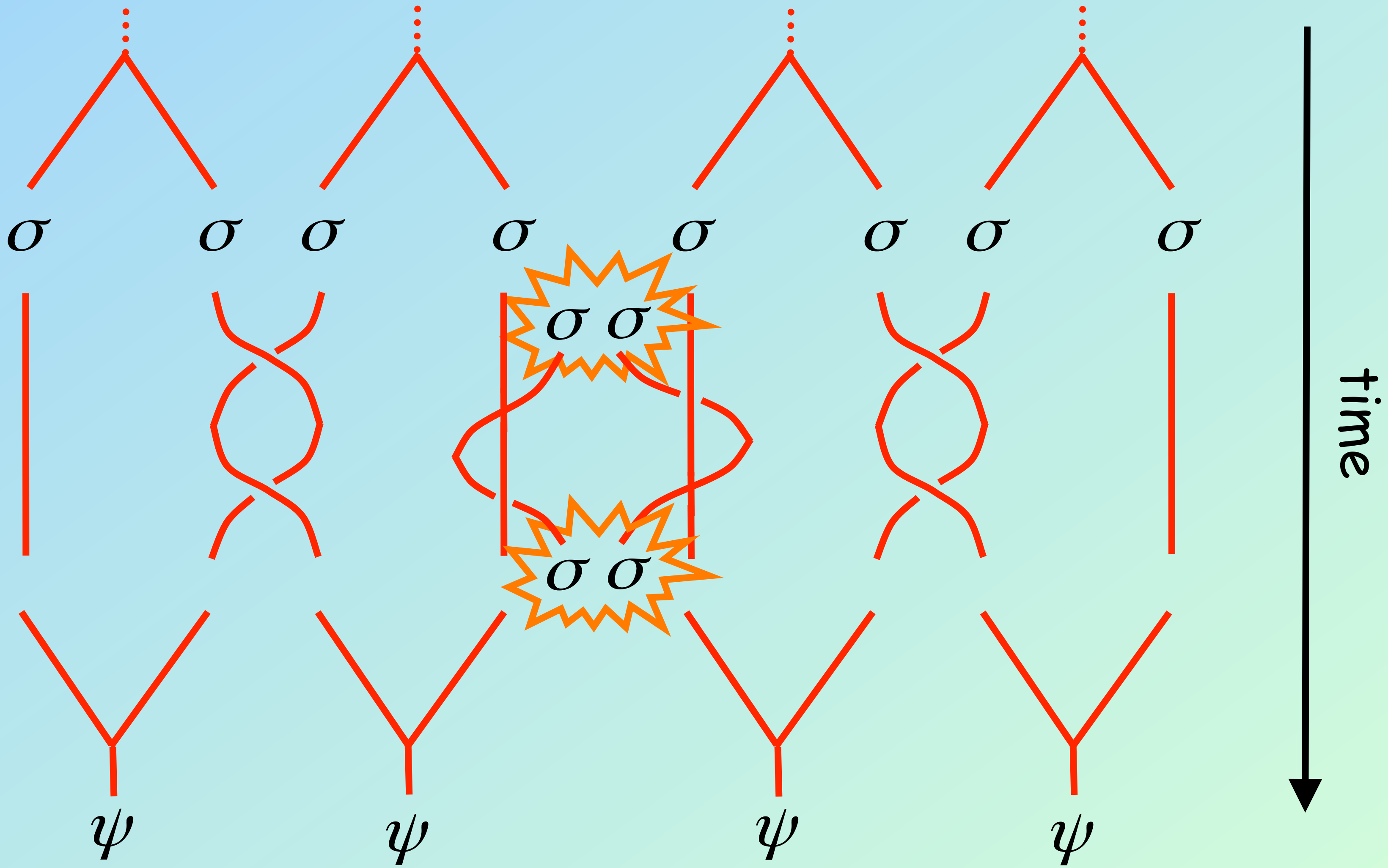
- Topological entropy:

$$\gamma = -\frac{1}{2}\left[(S_A - S_B) - (S_C - S_D)\right]$$

- Only loop contributions survive! [Levin & Wen]

Topological Errors

Errors can appear in the form of **virtual** anyons:

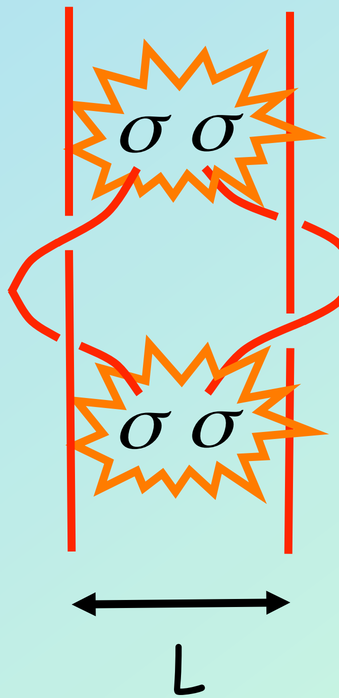


Topological Errors

Errors can appear in the form of virtual anyons

They can be avoided by keeping data anyons far apart:

$$P_{error} \sim e^{-\Delta L/v}$$



Δ : Energy Gap for σ pair cration

L : distance between σ anyons

v : characteristic velocity of anyons

Resilience to Errors

- **Abandon** the idea of separate **subsystems** for qubits. Encode info in **macroscopic degree of freedom** (non-locally).

  Direct observation of anyons does not reveal their total state.

  => local decoherence (environment "measures") does not destroy information.
- The **unitary transformations** resulting from braiding are virtually **errorless**.

Resilience to Errors

- Hamiltonian (energy gap) protects against local perturbations.
- Error corr ... against environmentally induced e ...

Topologically inspired quantum error correction.

0.75% tolerance [Raussendorf & Harrington]

Topological Memory

Can you create a 2D system that **resists errors** due to temperature for *long times*?

1) Toric code coupled to bosonic field:

errors (anyons) **attract** and annihilate!
[Hamma, Castelnovo & Chamon]

2) Induce a **repulsion** between anyons:

it generates a stable anyonic phase.
[Chesi, Roethlisberger & Loss]

3) *Entropic* **energy barriers**

[Brown, Al-Shimary, JKP]

Outlook

- **Quantum information** has a lot to offer to the study of topological systems.
- **Topological quantum computation** is a very promising way of storing and manipulating quantum information.
- Research on topological quantum computation has **applications** to many relevant fields of condensed matter, statistical physics, biology,...
- Topological states of matter NEED mathematics to be understood.