Knots, computation and materials

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Introduction
Topo Systems
Jones polynomials
Anyons
Ising & Fibonacci

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF LEEDS

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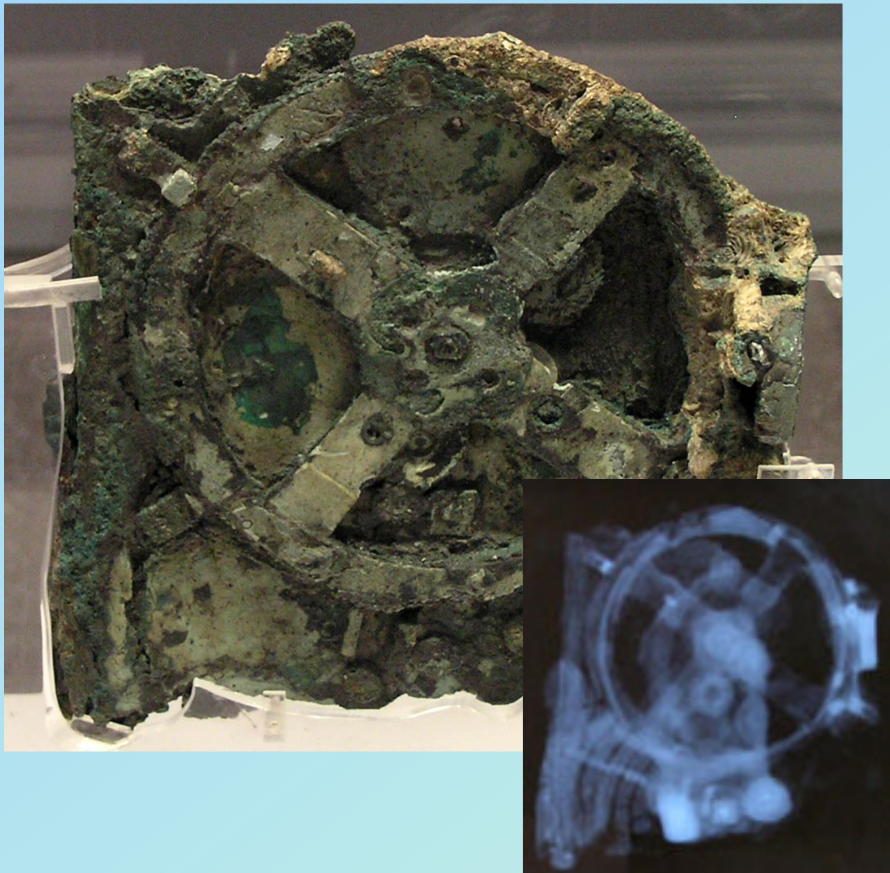
Computers

Antikythera mechanism



Robotron Z 9001

Analogue computer

Digital computer: 0 & 1

Quantum computers: Why?

- **Computational complexity**

  Problems that can be solved in:

  -polynomial time (easy)

  -exponential time (hard)

  as a function of input size.

- Classical computers:

  P: polynomially easy to solve

  NP: polynomially easy to verify solution

- **BQP**: polynomially easy to solve with QC

Quantum computers: Why?

- **Factoring**

  18070820886874048059516561644059055662781025167694013491701270214
  50056662540244048387341127590812303371781887966563182013214880557
  =
  39685999459597454290161126162883786067576449112810064832555157243
  ×
  45534498646735972188403686897274408864356301263205069600999044599

  quantum hackers *exponentially* better than classical hackers!

- **Searching** objects: where is ❥?

  ¢®¶ⓘ⦿🡇♫IΠÃ≥⅙⃠⏎✜➚✍🕔Ψ”✎👓♈✪❥⅖ű

- **Errors** during QC are **too catastrophic**.

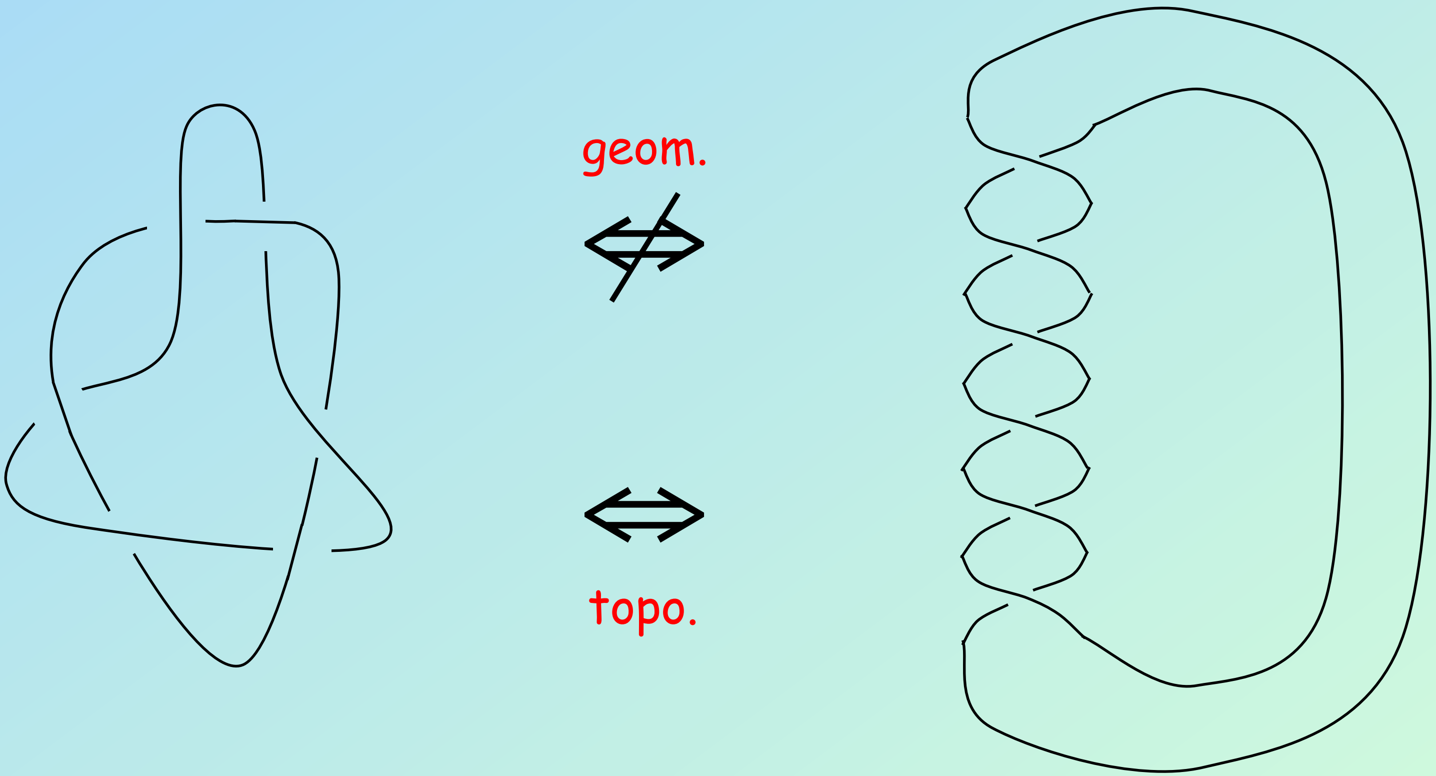
Topological quantum computers: Why?

Topology promises to solve the problem of **errors** that inhibit the experimental **realisation of quantum computers**…

…and it is a lot of fun :-)

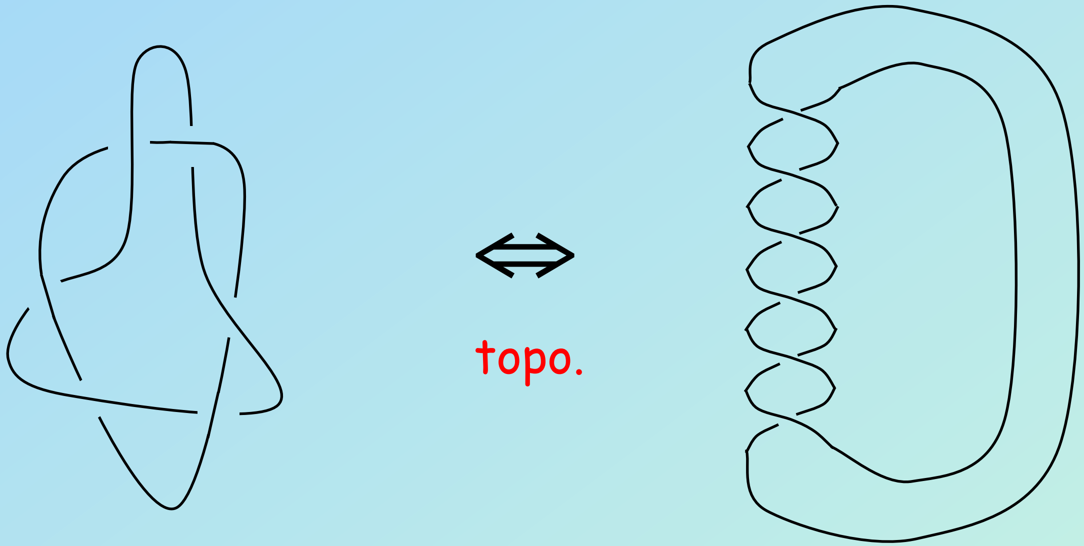
Geometry – Topology

- Geometry
  - Local properties of object
- Topology
  - Global properties of object



Topology of knots and links

Are two knots equivalent?



- Algorithms exist from the '60s
- **Extremely** time consuming…
- Common problem (speech recognition, …)
- *Mathematically* ***Jones polynomials*** *can recognise if two knots are inequivalent.*

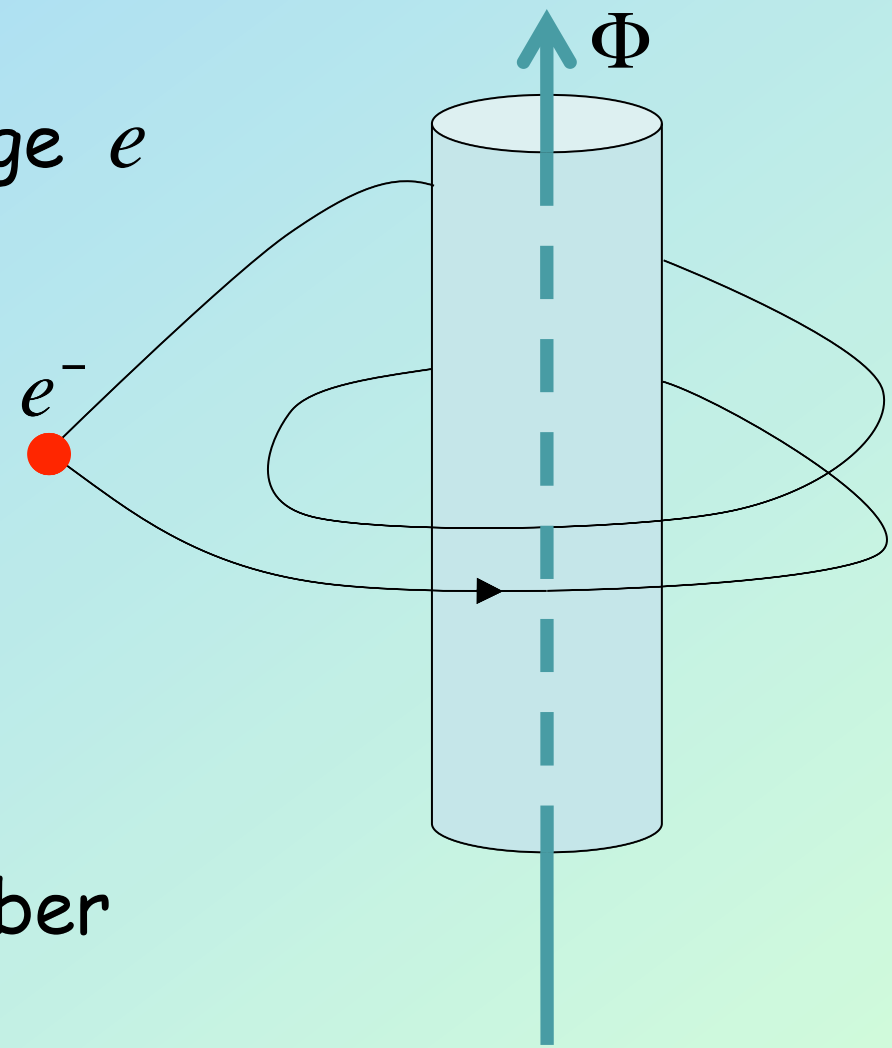
Topological quantum effects

Aharonov-Bohm effect

Magnetic flux Φ and charge e

$$\left|\Psi(\mathbf{x})\right\rangle \rightarrow e^{ine\Phi}\left|\Psi(\mathbf{x})\right\rangle$$

The phase is a function of winding number n

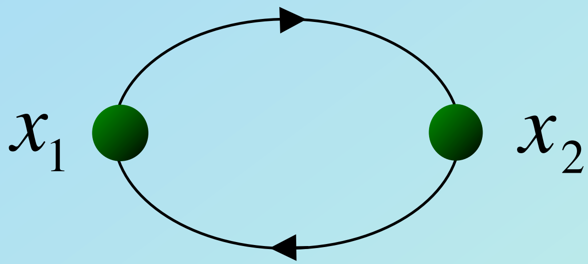


Topological effect:

n is the integer number of rotations

Particle statistics

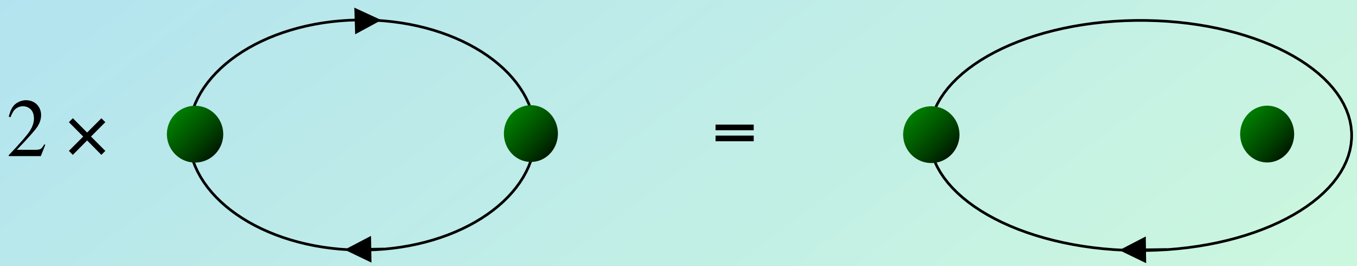
Exchange two **identical** particles:



Statistical symmetry:

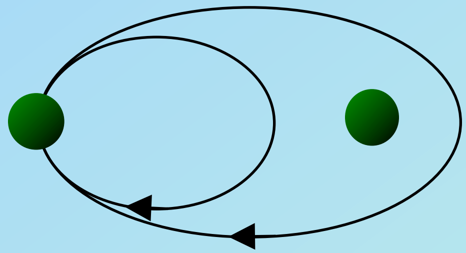
Physics stays the same, but $|\Psi\rangle$ could change!

$$|\Psi(x_1, x_2)\rangle = ???|\Psi(x_2, x_1)\rangle$$



Anyons and statistics

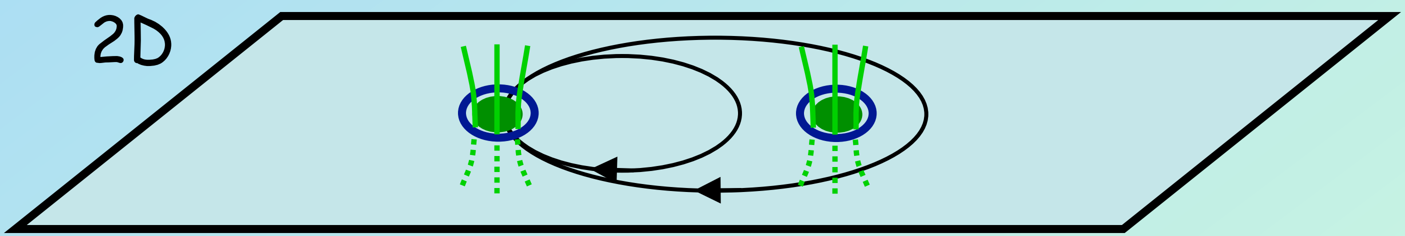
3D



Bosons $|\Psi\rangle \rightarrow |\Psi\rangle$

Fermions $|\Psi\rangle \rightarrow e^{i2\pi}|\Psi\rangle$

2D



$|\Psi\rangle \rightarrow e^{i2\phi}|\Psi\rangle$

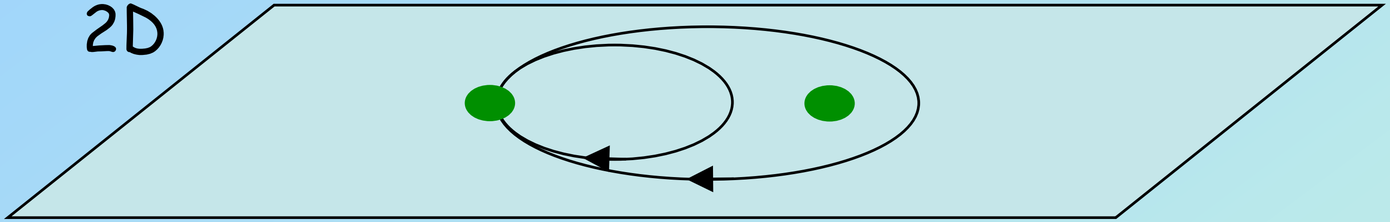
$|\Psi\rangle \rightarrow B|\Psi\rangle$

Anyons

Anyons: vortices with **flux** & **charge (fractional)**.

Aharonov-Bohm effect ⇔ **Berry Phase**.

Anyons and physical systems

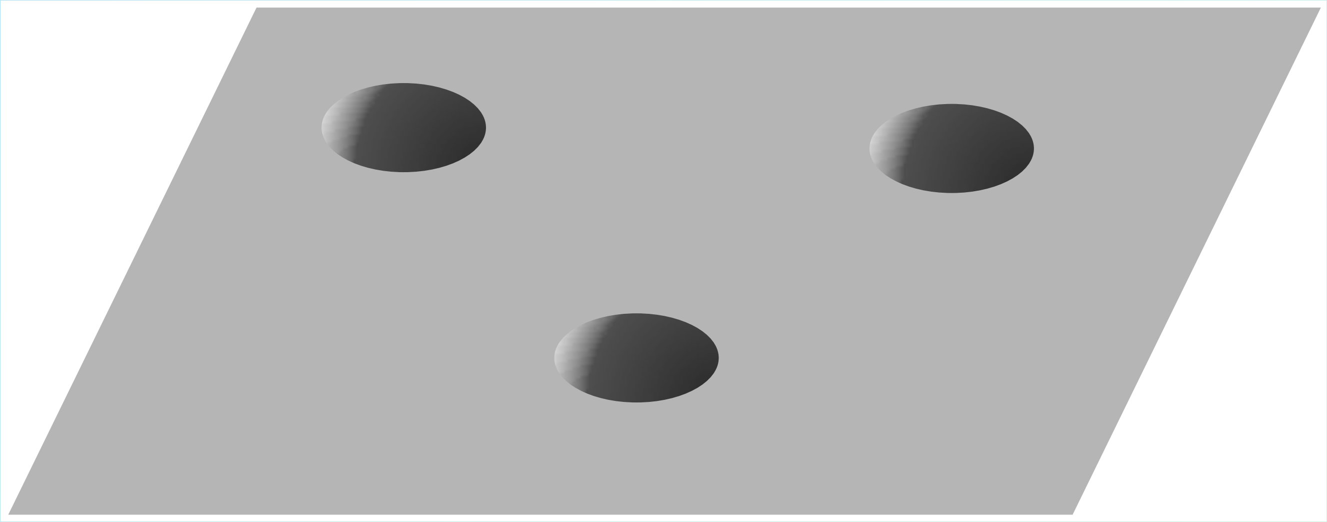


$$|\Psi\rangle \rightarrow e^{i2\phi}|\Psi\rangle$$

$$|\Psi\rangle \rightarrow B|\Psi\rangle$$

Anyons

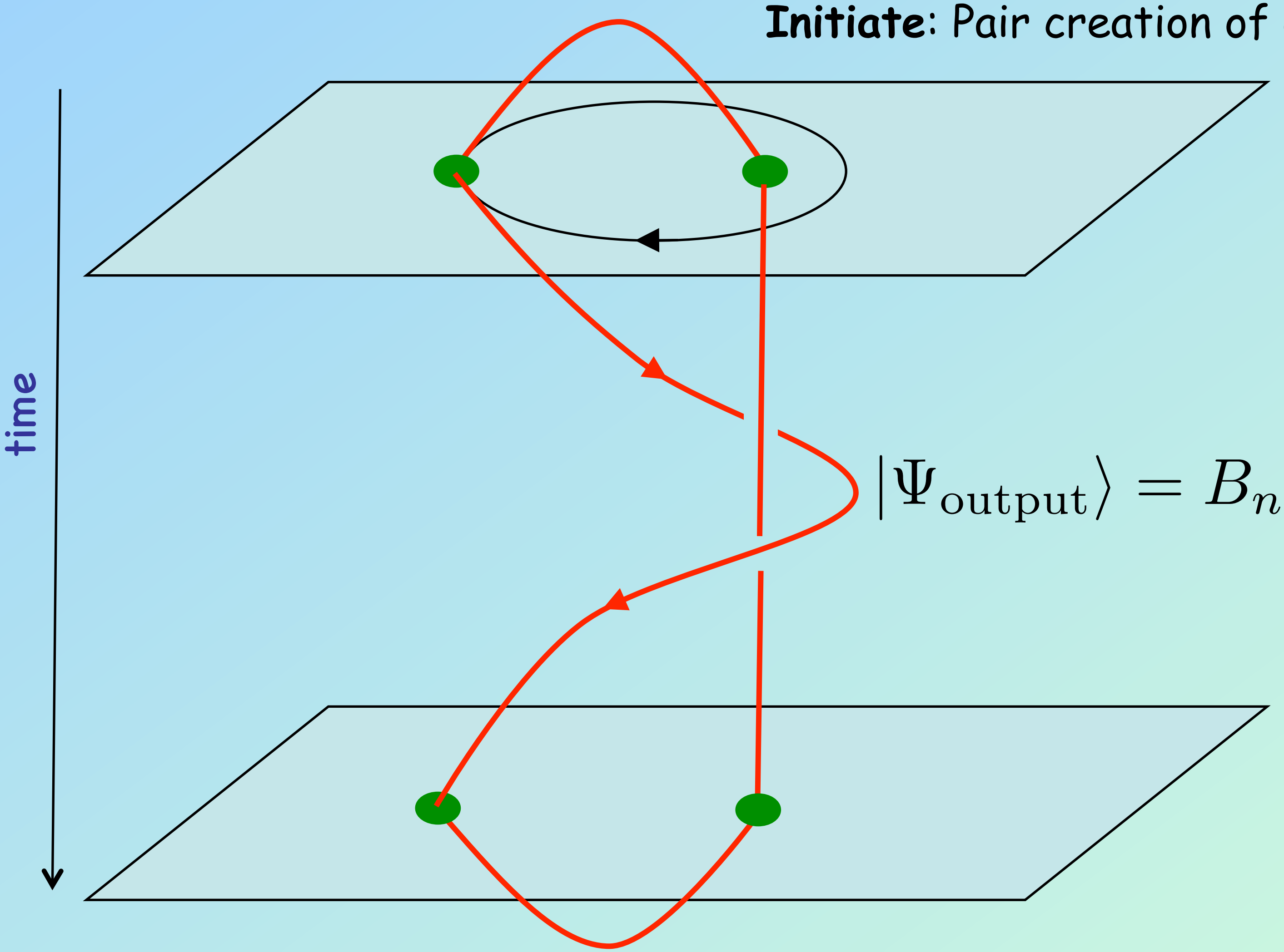
Anyonic properties can be found in 2-dimensional topological physical systems:



[G. Palumbo & JKP, "C-S from lattice", PRL 2013]

Anyons, statistics and knots

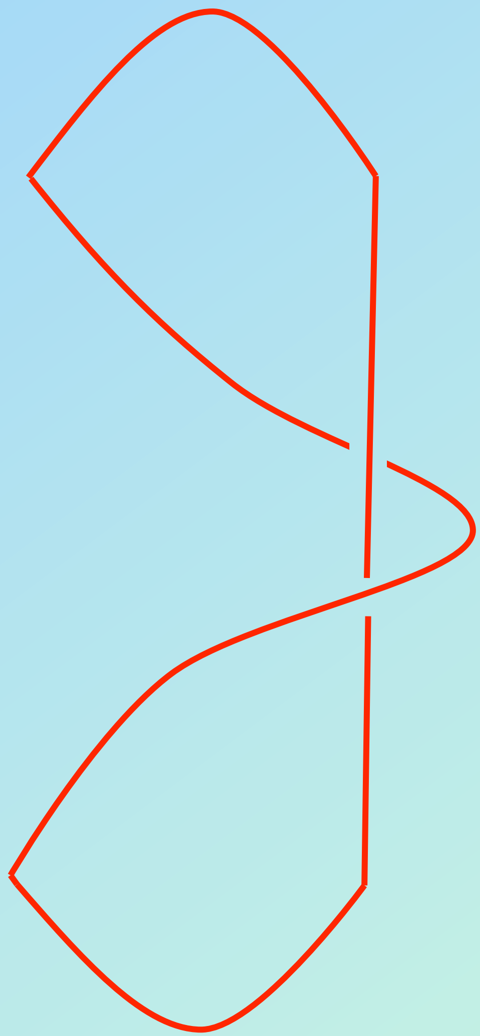
Initiate: Pair creation of anyons



$$|\Psi_{\mathrm{output}}\rangle = B_n ... B_2 B_1 |\Psi_{\mathrm{input}}\rangle$$

Measure: do they fuse to the vacuum?

Anyons, statistics and knots



$$|\Psi_{\text{output}}\rangle = B_n \ldots B_2 B_1 |\Psi_{\text{input}}\rangle$$

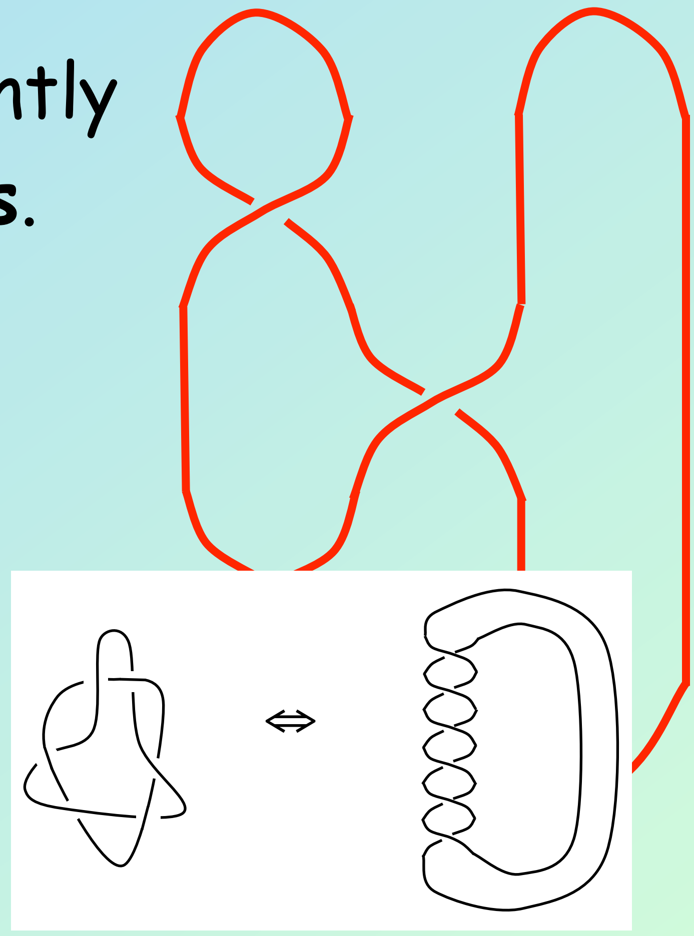
Anyons and knots

Assume I can generate anyons in the laboratory.

- The state of anyons is efficiently described by their **world lines**.

- Creation, braiding, fusion.

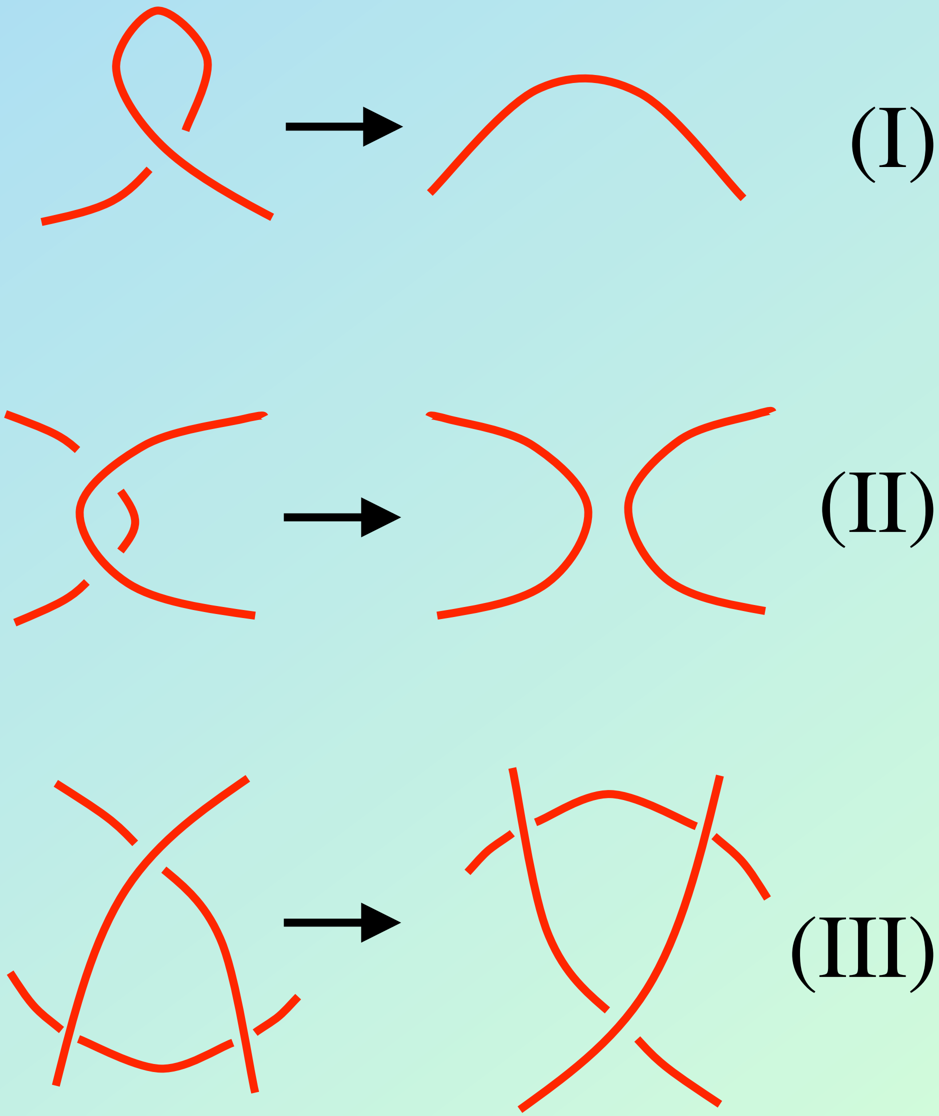
- The final **quantum state** of anyons is **invariant** under **continuous** deformations of strands.



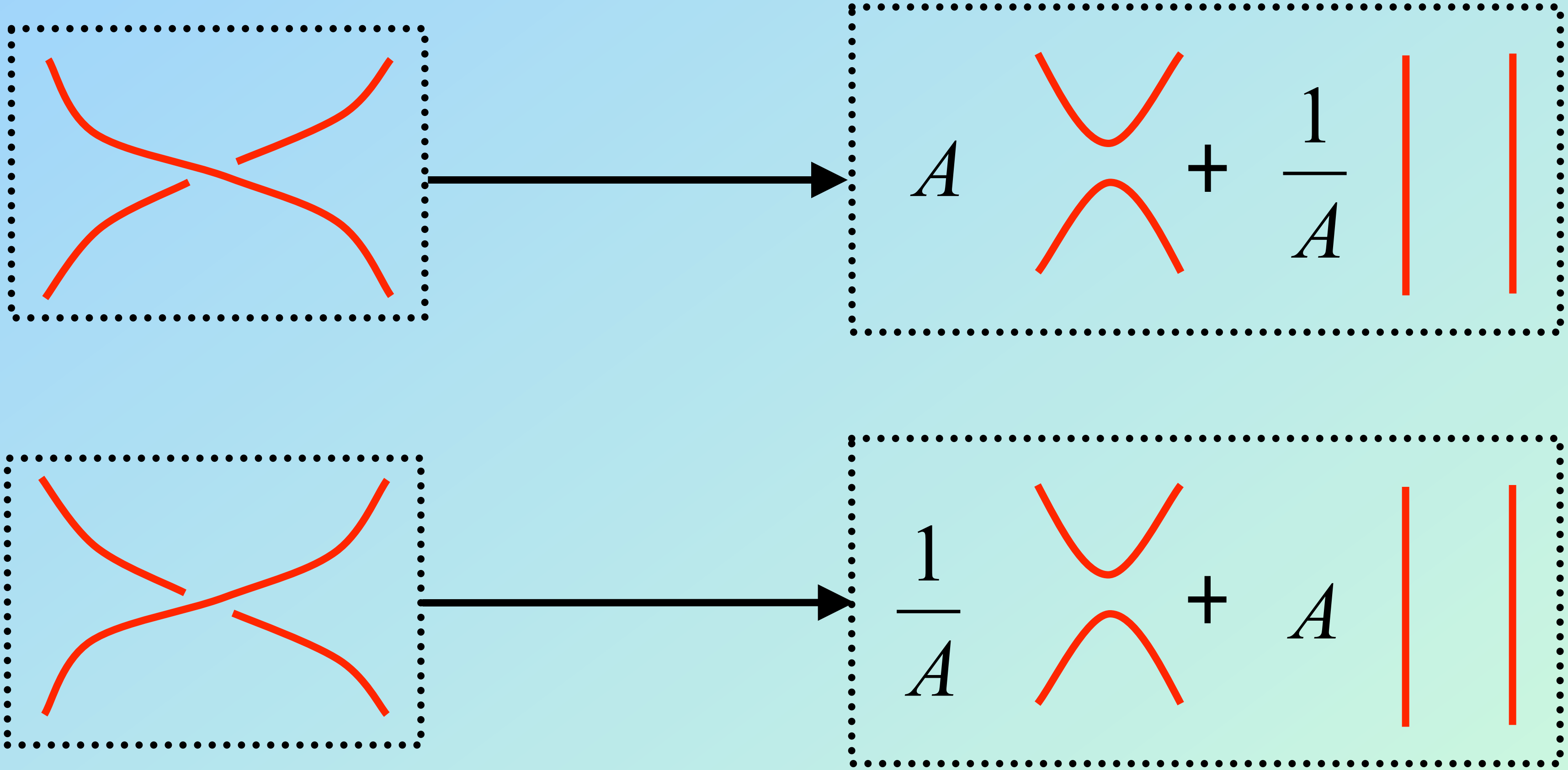
The Reidemeister moves

Theorem:

Two knots can be **deformed continuously** one into the other iff one knot can be transformed into the other by **local moves**:

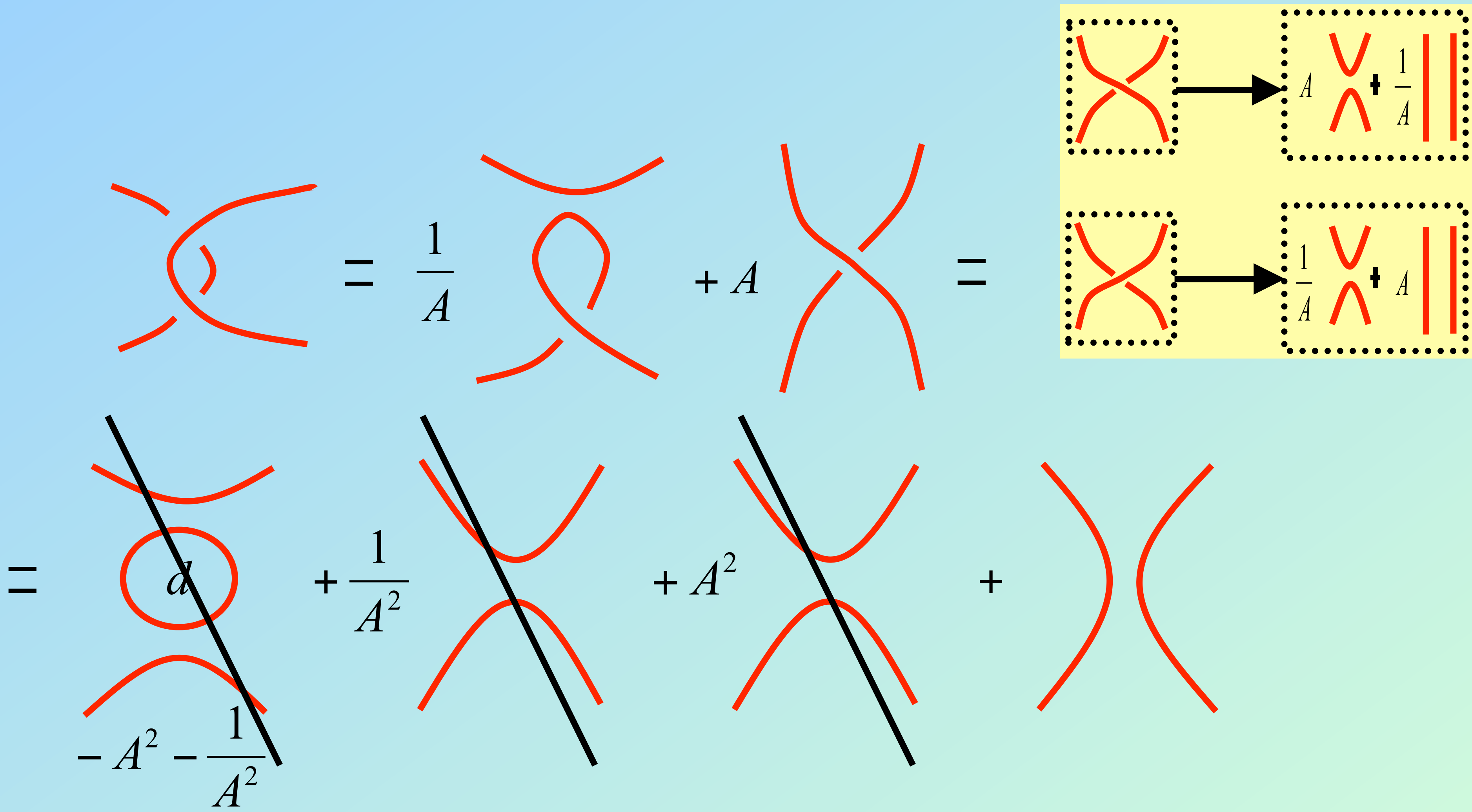


Skein relations



$$-A^2 - \frac{1}{A^2} = d$$

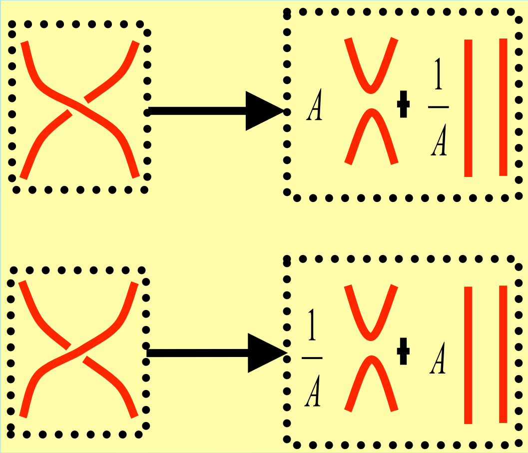
Skein and Reidemeister

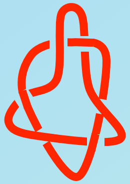


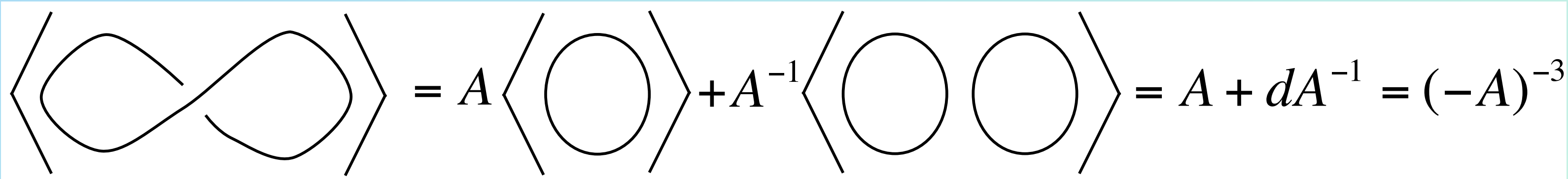
Reidemeister move (II) is satisfied. Similarly (III).

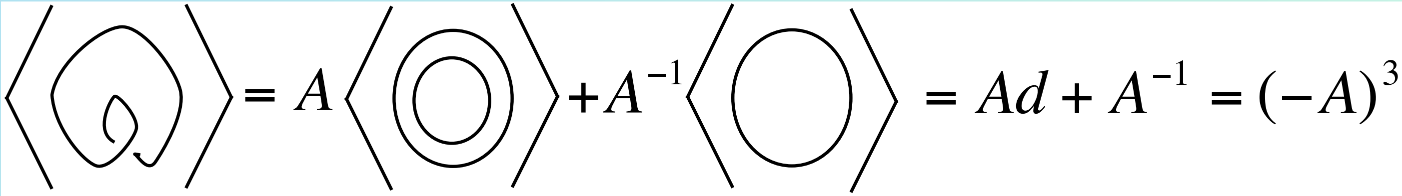
Kauffman bracket

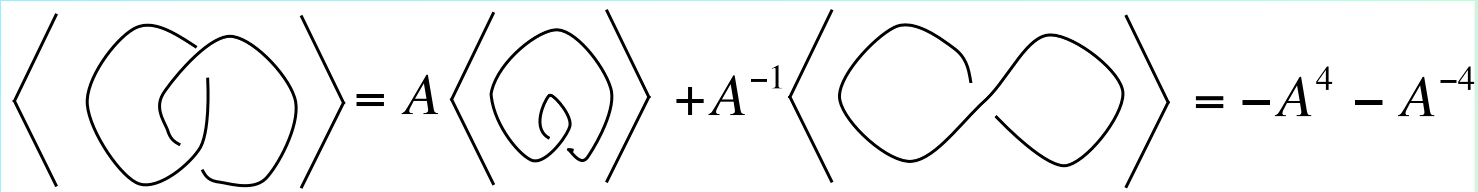
The Skein relations give rise to the **Kauffman bracket**:



Skein() $= \langle L \rangle (A)$







Jones polynomial

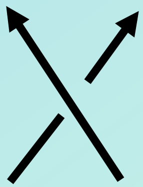
The Skein relations give rise to the **Kauffman bracket**:

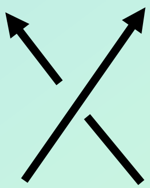
Skein()= $\langle L \rangle (A)$

To satisfy **move (I)** one needs to define **Jones polynomial**:

$$V_L(A) = (-A)^{3w(L)} \langle L \rangle (A)$$

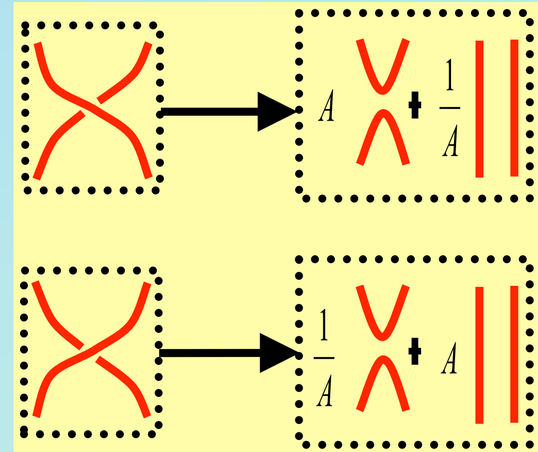
$w(L)$ is the **writhe** of link. For an oriented link it is the sum of the signs for all crossings

 $= +1$

 $= -1$

Jones polynomial

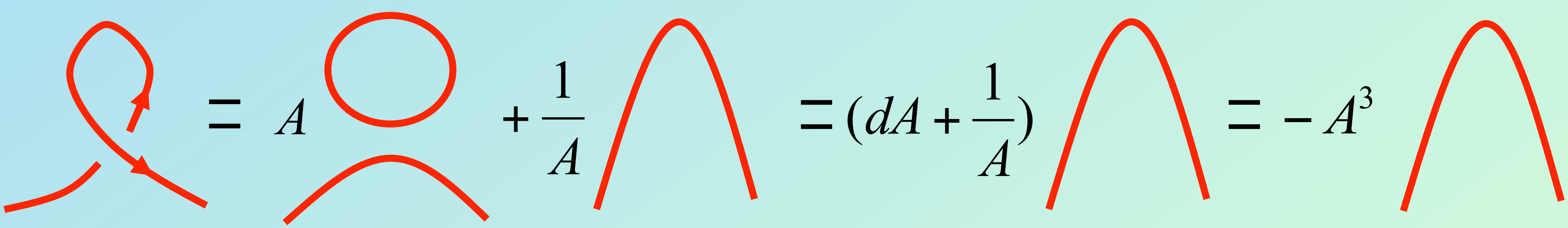
The Skein relations give rise to the **Kauffman bracket**:



Skein() $=\langle L\rangle(A)$

To satisfy **move (I)** one needs to define **Jones polynomial**:

$$V_L(A)=(-A)^{3w(L)}\langle L\rangle(A)$$



$w=-1$

Jones polynomial

- If two links have different Jones polynomials then they are inequivalent

=> **use it to distinguish links**

- Jones polynomials keep:

only topological information, **no geometrical**

Jones polynomial from anyons

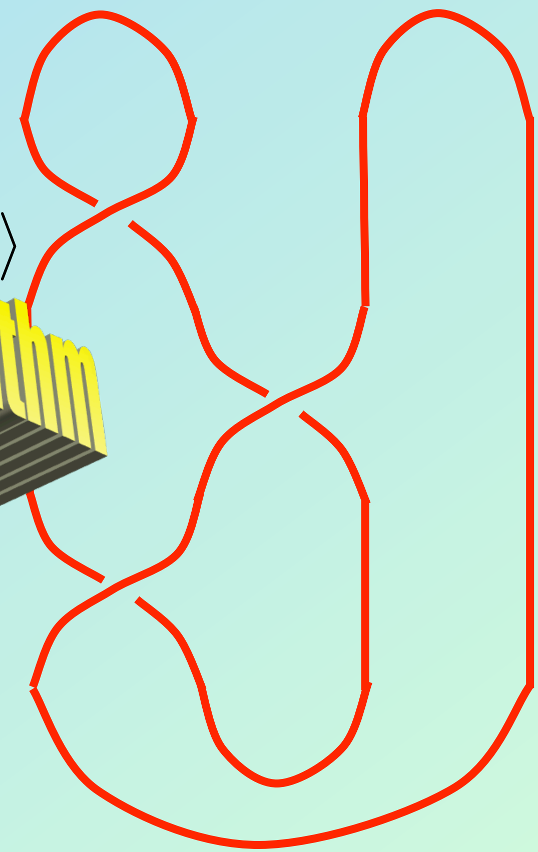
Braiding evolutions of anyonic states:

$$|\Psi_{\text{final}}\rangle = \ldots B_2 B_1 |\Psi_{\text{initial}}\rangle$$

$$\langle\Psi_{\text{initial}}|\Psi_{\text{final}}\rangle = \ldots |\Psi_{\text{initial}}\rangle$$

$$= \frac{1}{d^{n/2-1}} \langle L(B) \rangle$$

A new quantum algorithm



- *Simulate the knot with braiding anyons*
- Translate it to circuit model:
  <=> find trace of matrices

Jones polynomial from QC

Evaluating Jones polynomials is a #P-hard problem.

Belongs to **BQP class**.

With quantum computers it is **polynomially** easy to *approximate* with additive error.

[Freedman, Kitaev, Larsen, Wang (2002);
Aharonov, Jones, Landau (2005);
Kauffman, Glaser *et al.* (2009);
Kuperberg (2009)]

Summary

Jones polynomials are used for quantum applications:

•**encrypt** quantum information

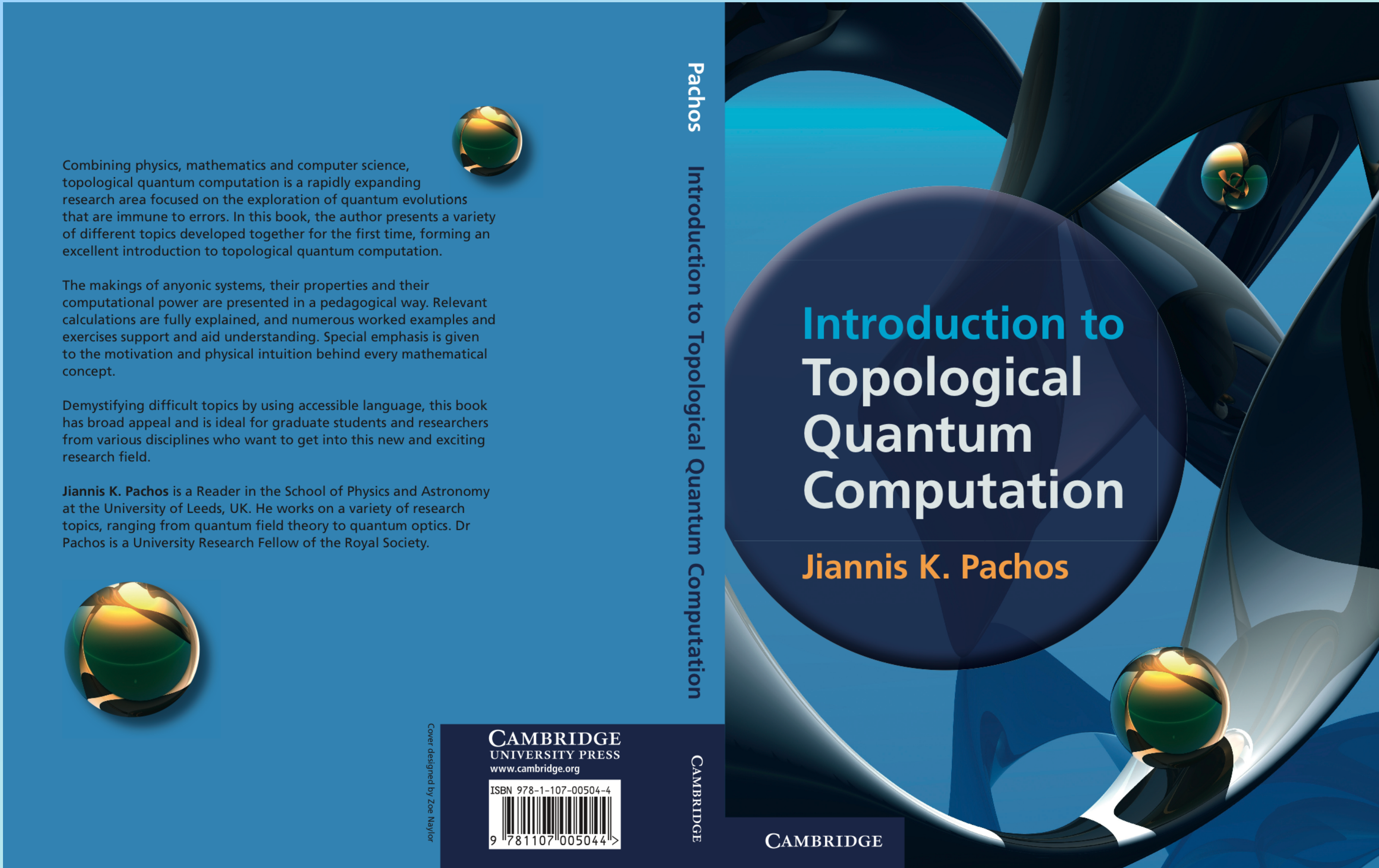
•**quantum money**

•…

Topological systems that can support **anyons** are currently **engineered**…

http://quantum.leeds.ac.uk/~jiannis

Book


Combining physics, mathematics and computer science, topological quantum computation is a rapidly expanding research area focused on the exploration of quantum evolutions that are immune to errors. In this book, the author presents a variety of different topics developed together for the first time, forming an excellent introduction to topological quantum computation.
The makings of anyonic systems, their properties and their computational power are presented in a pedagogical way. Relevant calculations are fully explained, and numerous worked examples and exercises support and aid understanding. Special emphasis is given to the motivation and physical intuition behind every mathematical concept.
Demystifying difficult topics by using accessible language, this book has broad appeal and is ideal for graduate students and researchers from various disciplines who want to get into this new and exciting research field.
Jiannis K. Pachos is a Reader in the School of Physics and Astronomy at the University of Leeds, UK. He works on a variety of research topics, ranging from quantum field theory to quantum optics. Dr Pachos is a University Research Fellow of the Royal Society.
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Jiannis K. Pachos
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for your great kindness in the matter of the names respecting which I applied to you; but I hoped to have met you last Saturday at Kensington and therefore delayed expressing my obligations

I have taken your advice and the names used are <u>anode</u> <u>cathode</u> <u>anions</u> <u>cations</u> and <u>ions</u> the last I shall have but little occasion for. I had some hot objections made to them here and found myself very much in the condition of the man with his son and Ass who tried to please every body; but when

Letter from Faraday to Whewell (1834)

Inception of Anyonic Models

1. Take a certain number of **different anyons**

   1, a, b, ...

   the vacuum (1) and one or more non-trivial particles

2. Define **fusion rules** between them

   1×a=a, a×b=c+d+..., a×a=1+...

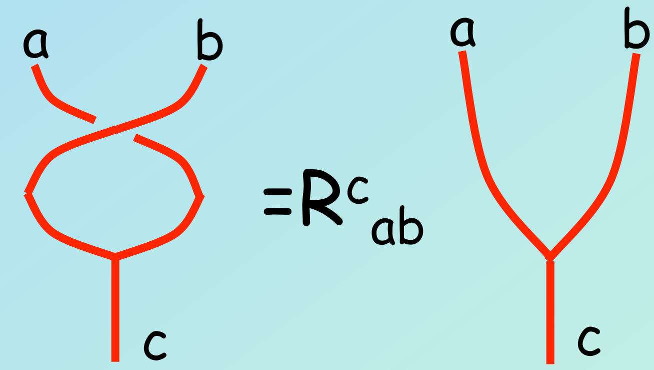
   The **vacuum** acts trivially. Each particle has an **anti-particle** (might be itself or not).

- Abelian anyons axb=c
- Non-Abelian anyons axb=c+d+...

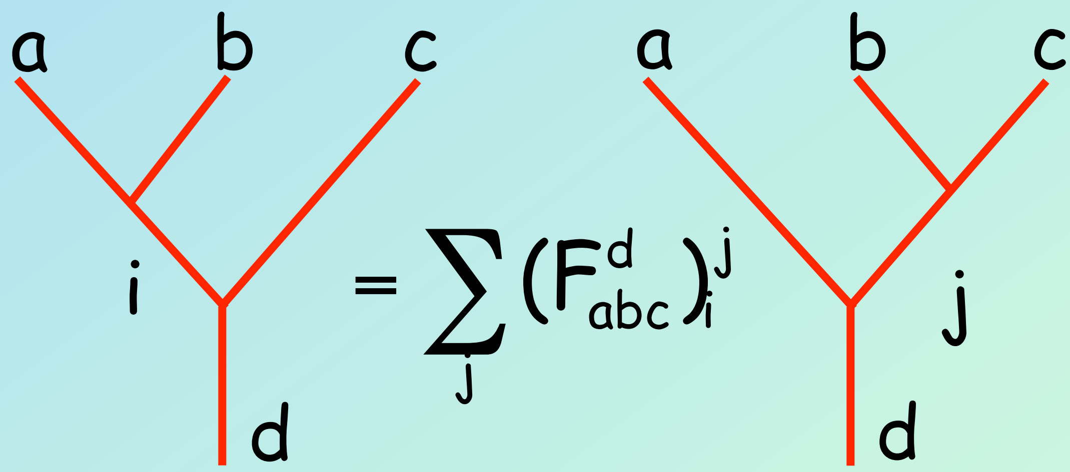
Braiding and Fusion properties

- The action of braiding of two anyons depends on their fusion outcome:

R^c_{ab} is a phase factor



- Changing the order of fusion is non-trivial:



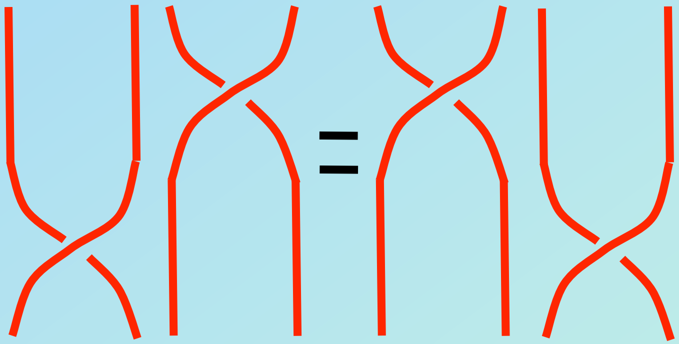
The braid group B_n

The braid group B_n has elements $b_1, b_2, \ldots, b_{n-1}$ that satisfy:

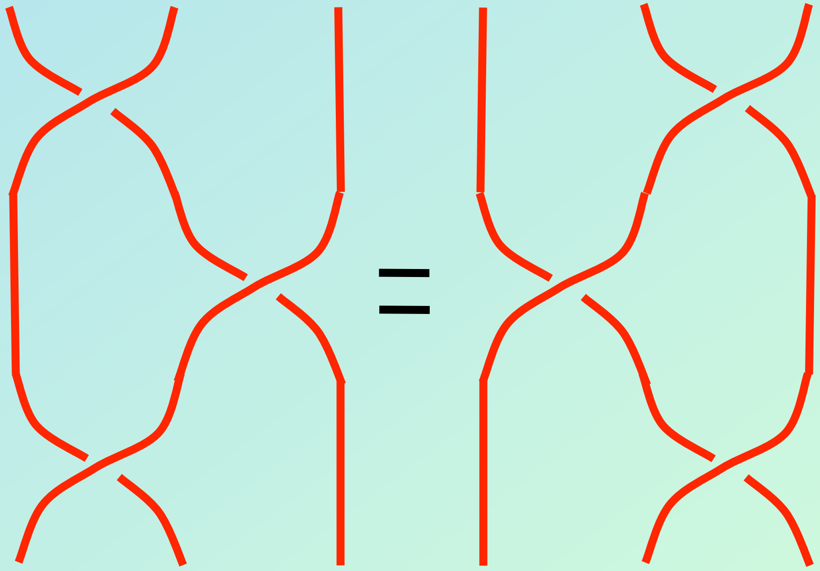
$$b_i b_j = b_j b_i, \ \ for\ |i-j| \geq 2$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \ \ for\ 1 \leq i < n$$

Pictorially:



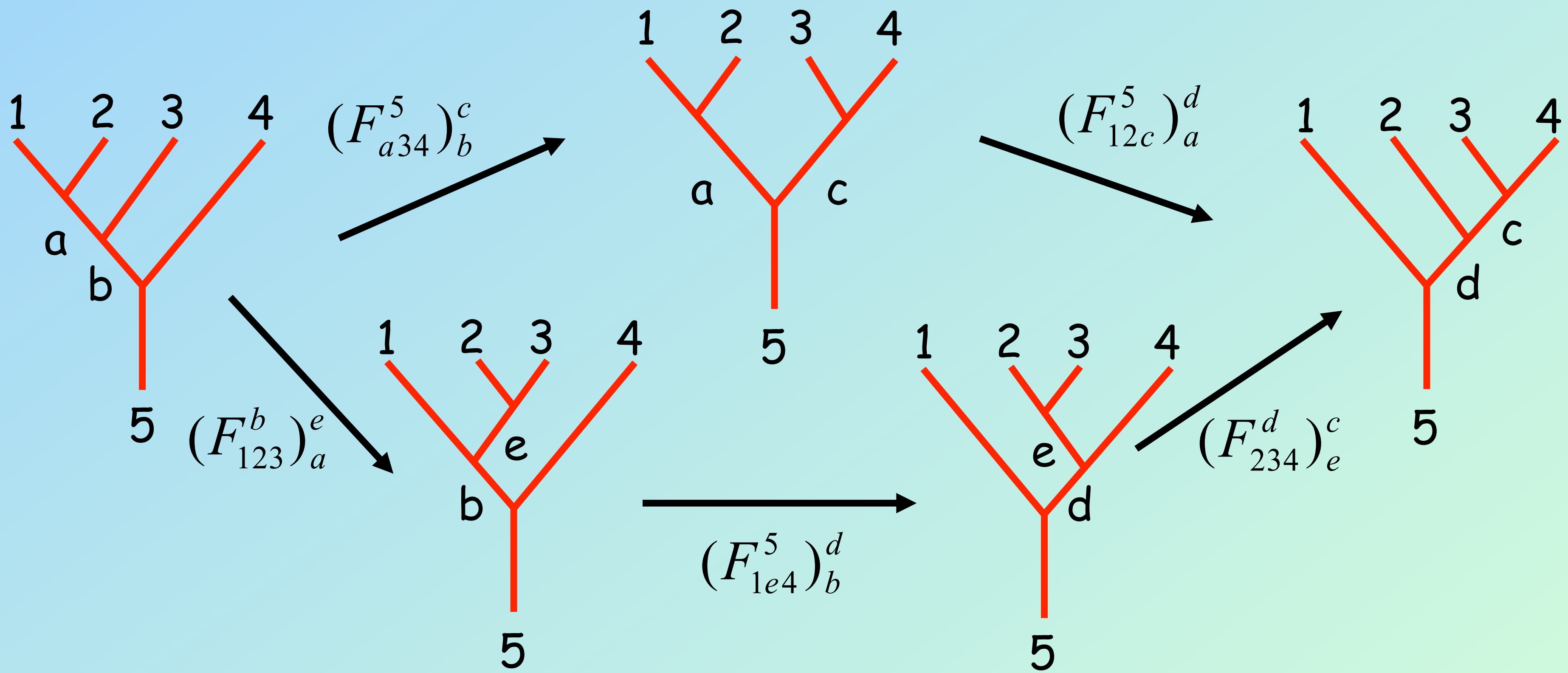
$$b_i b_j = b_j b_i$$



$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

Inception of Anyonic Models

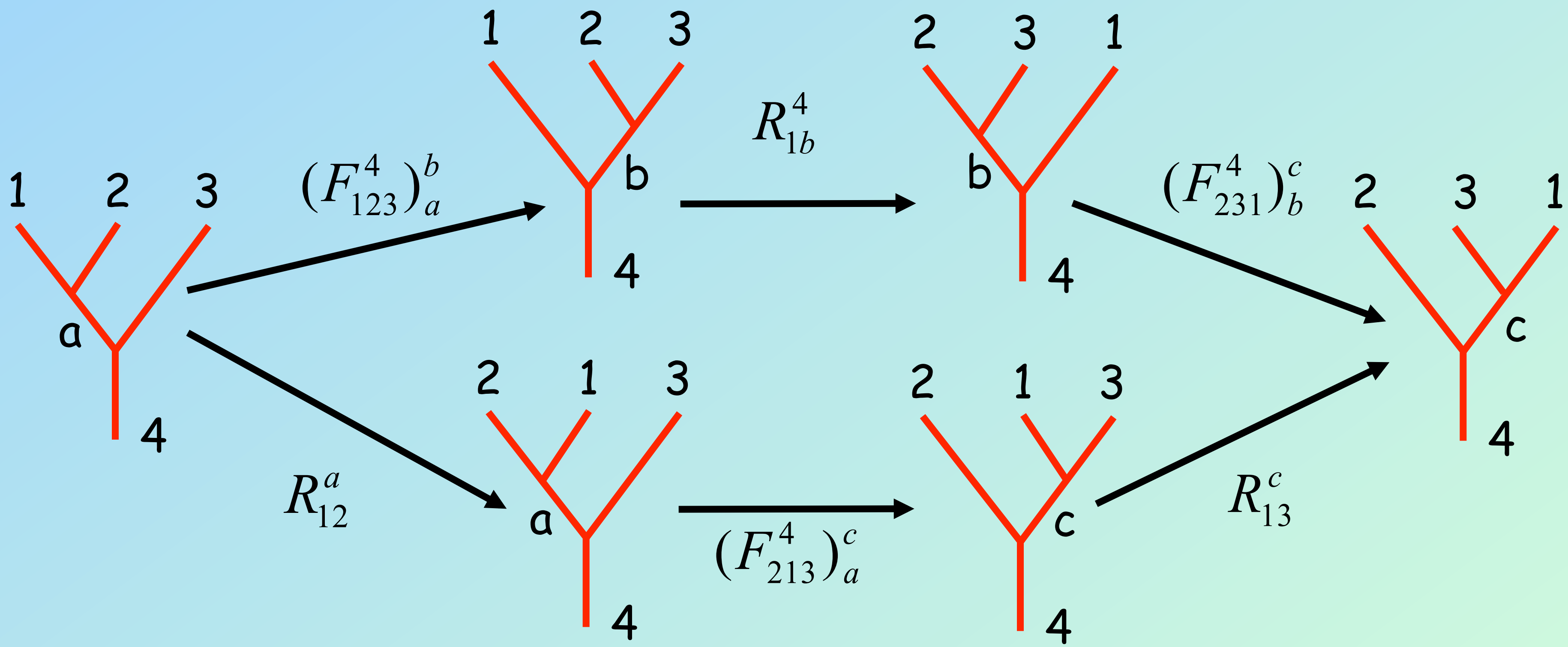
3. The F and B matrices are determined from the **Pentagon** and Hexagon identities



$$(F^5_{12c})^d_a (F^5_{a34})^c_b = \sum_e (F^d_{234})^c_e (F^5_{1e4})^d_b (F^b_{123})^e_a$$

Inception of Anyonic Models

3. The F and B matrices are determined from the Pentagon and **Hexagon** identities

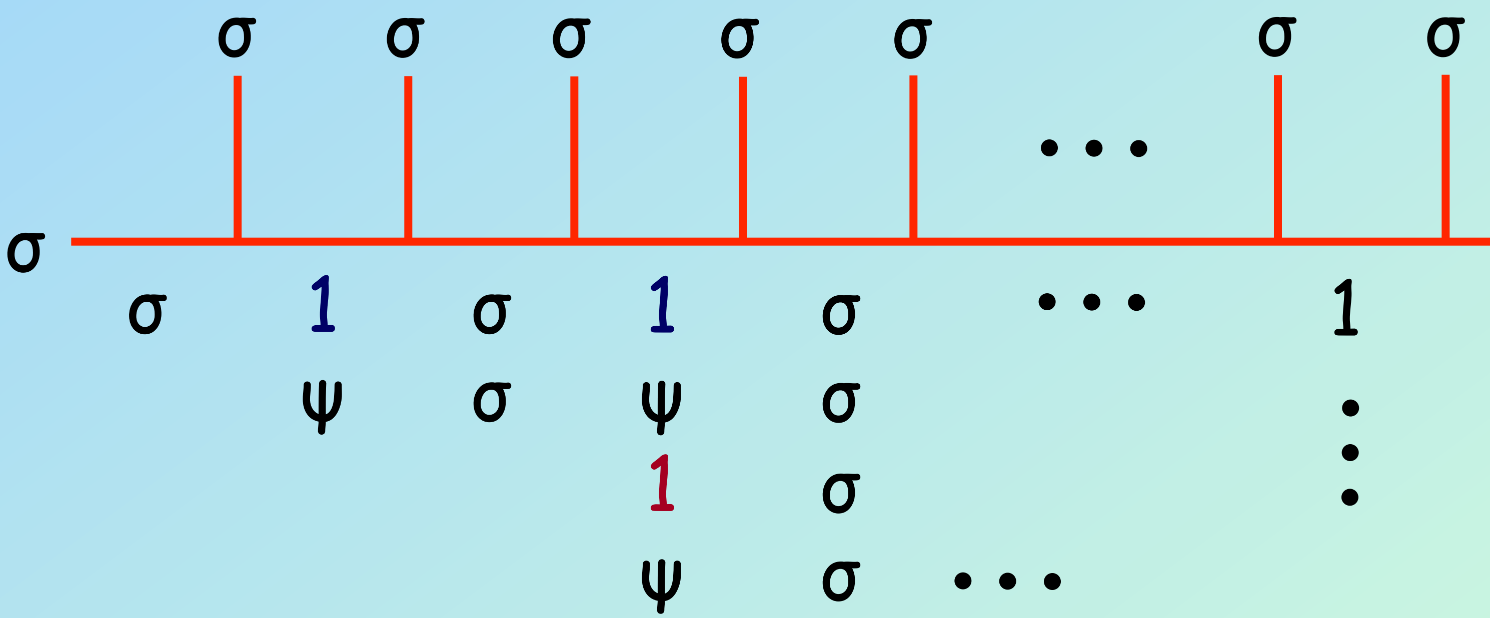


$$\sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a = R^c_{13} (F^4_{213})^c_a R^a_{12}$$

Ising Anyons

Consider the particles: 1, σ and ψ

Fusion rules: σ×σ=1+ψ, ψ×ψ=1, σ×ψ=σ



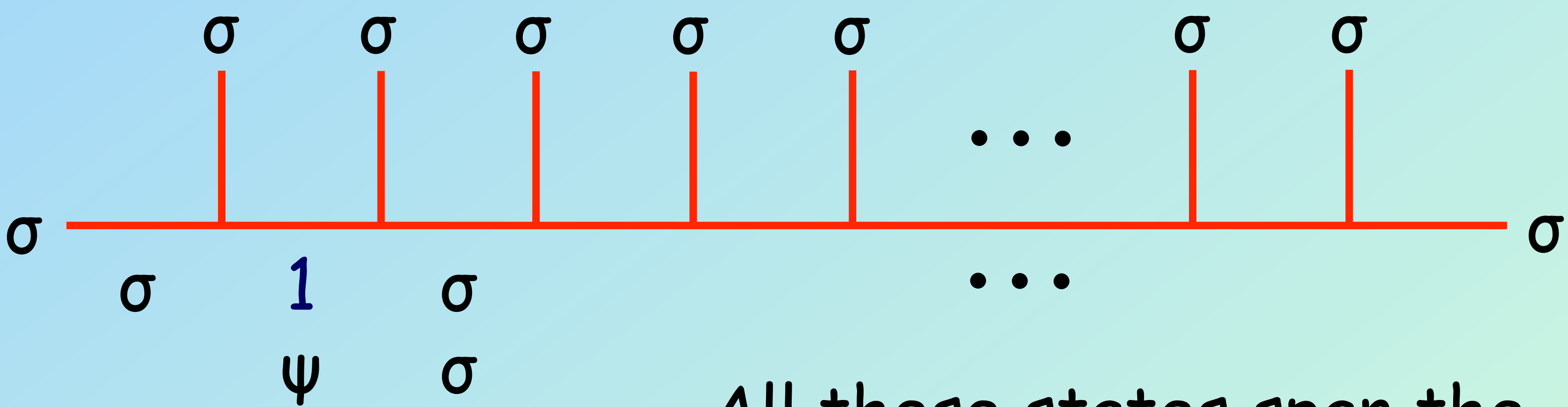
$d_2=2$
$d_4=4$
$d_6=8$
$d_8=16$
…

$d_n=2^{n/2}$ increase in dim of Hilbert space

Ising Anyons

Consider the particles: 1, σ and ψ

Fusion rules: $\sigma\times\sigma=1+\psi$, $\psi\times\psi=1$, $\sigma\times\psi=\sigma$



$$|\Psi\rangle = |1,1,\ldots\rangle$$

$$|\Psi\rangle = |1,\psi,\ldots\rangle$$

All these states span the **fusion Hilbert space**.

Braiding neighboring anyons transforms states

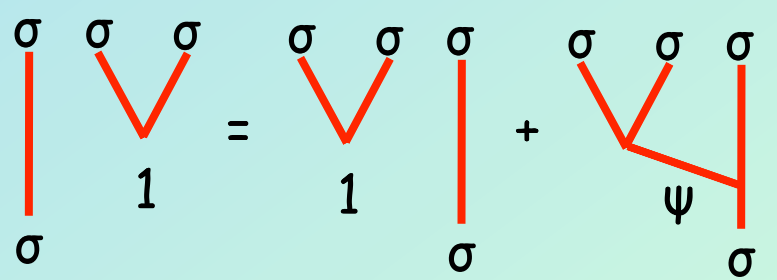
Ising Anyons

Consider the particles: 1, σ and ψ

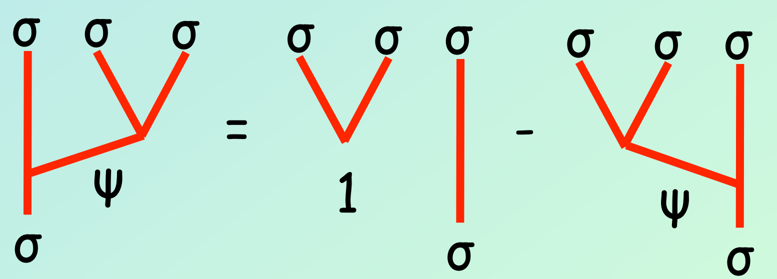
Fusion rules: $\sigma\times\sigma=1+\psi$, $\psi\times\psi=1$, $\sigma\times\psi=\sigma$

From 5-gon and 6-gon identities we have:

$$F^{\sigma}_{\sigma\sigma\sigma}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=H$$

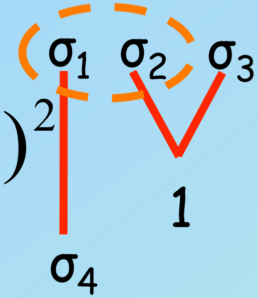


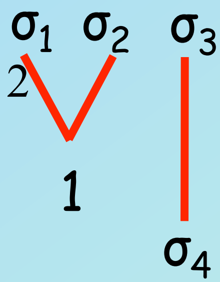
Rotation of basis states

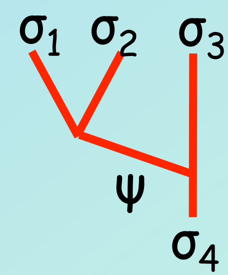


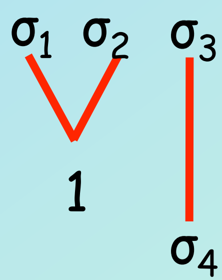
Ising Anyons

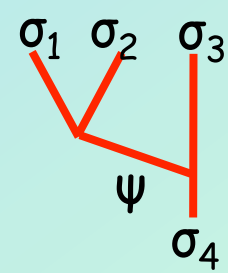
Braiding $R^{1}_{\sigma\sigma} = e^{-i\pi/8}$ and $R^{\psi}_{\sigma\sigma} = ie^{-i\pi/8}$ $\Rightarrow R_{\sigma\sigma} = e^{-i\pi/8}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

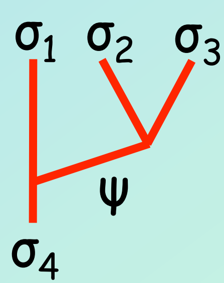
$(R_{\sigma_1\sigma_2})^2$ 

$= (R^{1}_{\sigma_1\sigma_2})^2$ 

$+ (R^{\psi}_{\sigma_1\sigma_2})^2$ 

$= e^{-i\pi/4}$ 

$- e^{-i\pi/4}$ 

$= e^{-i\pi/4}$ 

$H\sigma^z H = \sigma^x$

Clifford group: non-universal!

Ising Anyons

Qubit initialization:

State |0> State |1>

Measurement: Outcome of pairwise fusion, 1 or ψ

$$H\sigma^z H = \sigma^x$$

Gates: Clifford group. Non-universal!
One needs a **phase gate**: employ interactions between anyons.

Can be employed as a quantum memory.

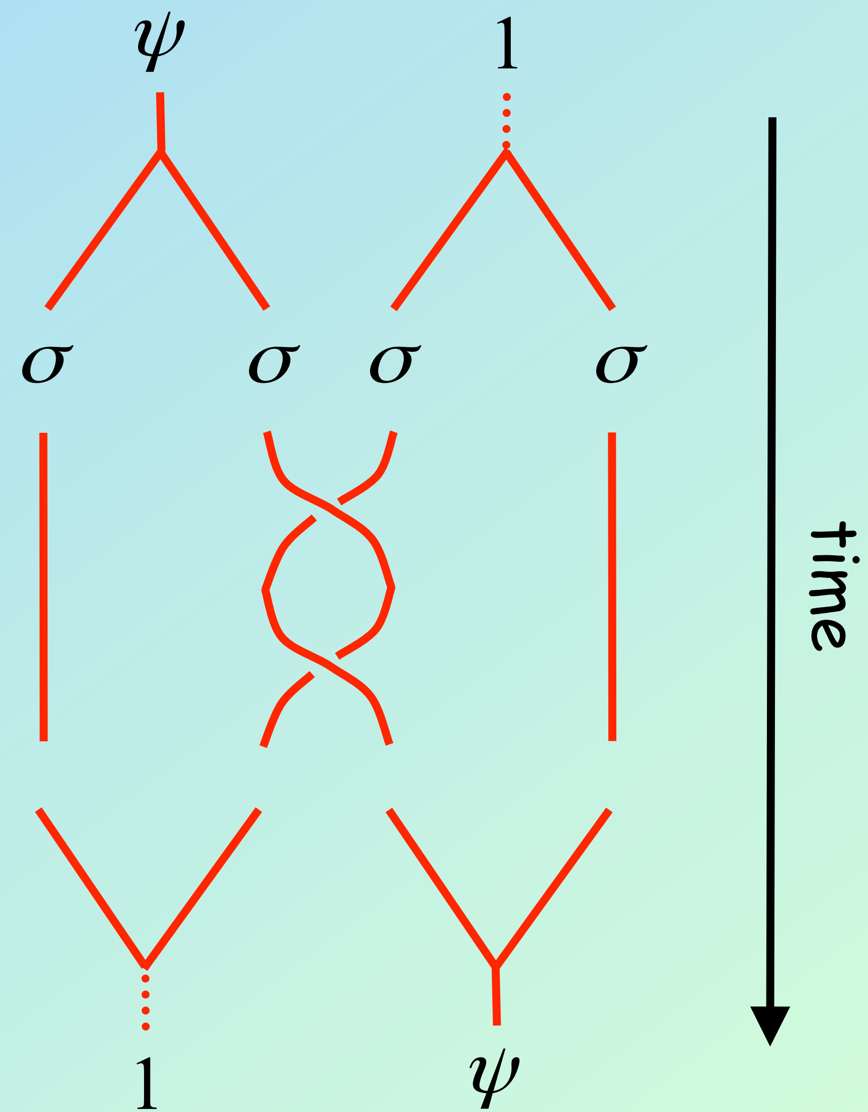
Ising Anyons

- Assume we can:
  - **Create** identifiable anyons
    vacuum pair creation
  - **Braid** anyons
    Statistical evolution:
    braid representation B
  - **Fuse** anyons

$$\sigma \times \sigma = 1 + \psi$$

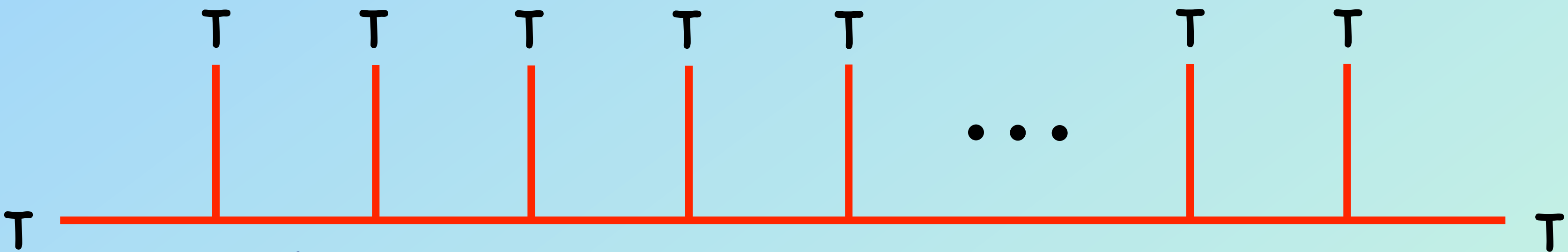
Fusion Hilbert space:

$$|\sigma, \sigma \rightarrow 1\rangle, |\sigma, \sigma \rightarrow \psi\rangle$$



Fibonacci Anyons

Consider anyons with labels 1 or τ with the fusion properties: **1×1=1, 1×τ = τ, τ×τ =1+τ**



τ 1 τ 1 τ
τ 1 τ 1
τ τ τ
1 1 …
τ τ
τ
1
τ

$d_1=1$
$d_2=2$
$d_3=3$
$d_4=5$
$d_5=8$
…

Fibonacci sequence!

Dimension of Hilbert space

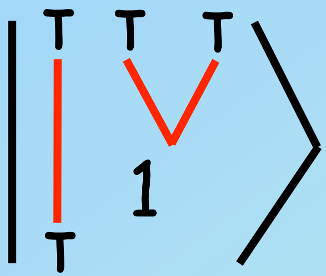
$$d_n \propto \phi^n$$

$$\phi = (1+\sqrt{5})/2$$

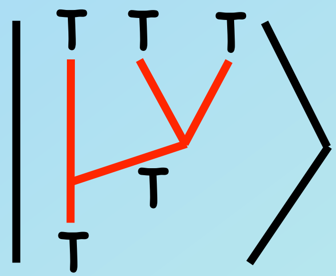
Golden mean

Fibonacci Anyons and QC

Qubit encoding:



State $|0\rangle$

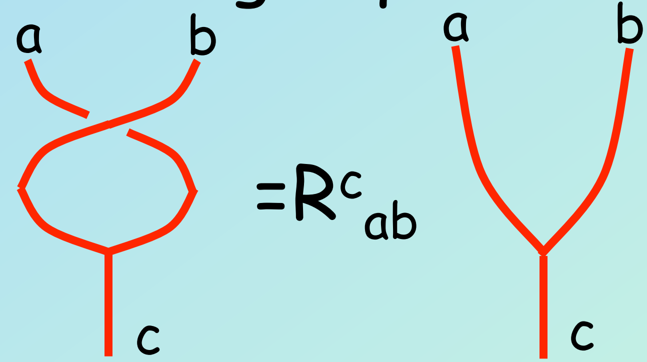


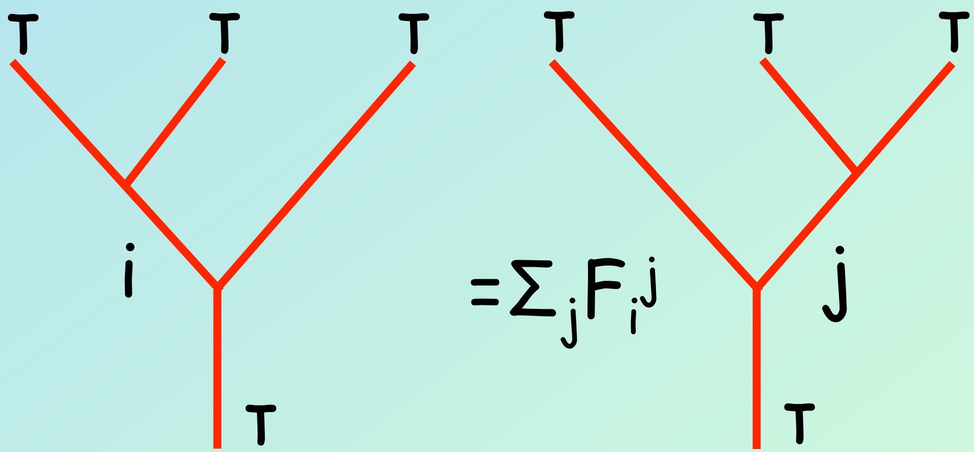
State $|1\rangle$

$$|\tau,\tau \rightarrow 1\rangle = |0\rangle$$

$$|\tau,\tau \rightarrow \tau\rangle = |1\rangle$$

Evolving a qubit:



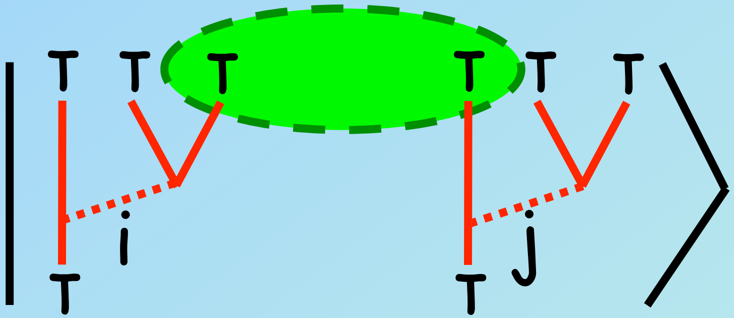


Unitaries B and F are dense in SU(2).

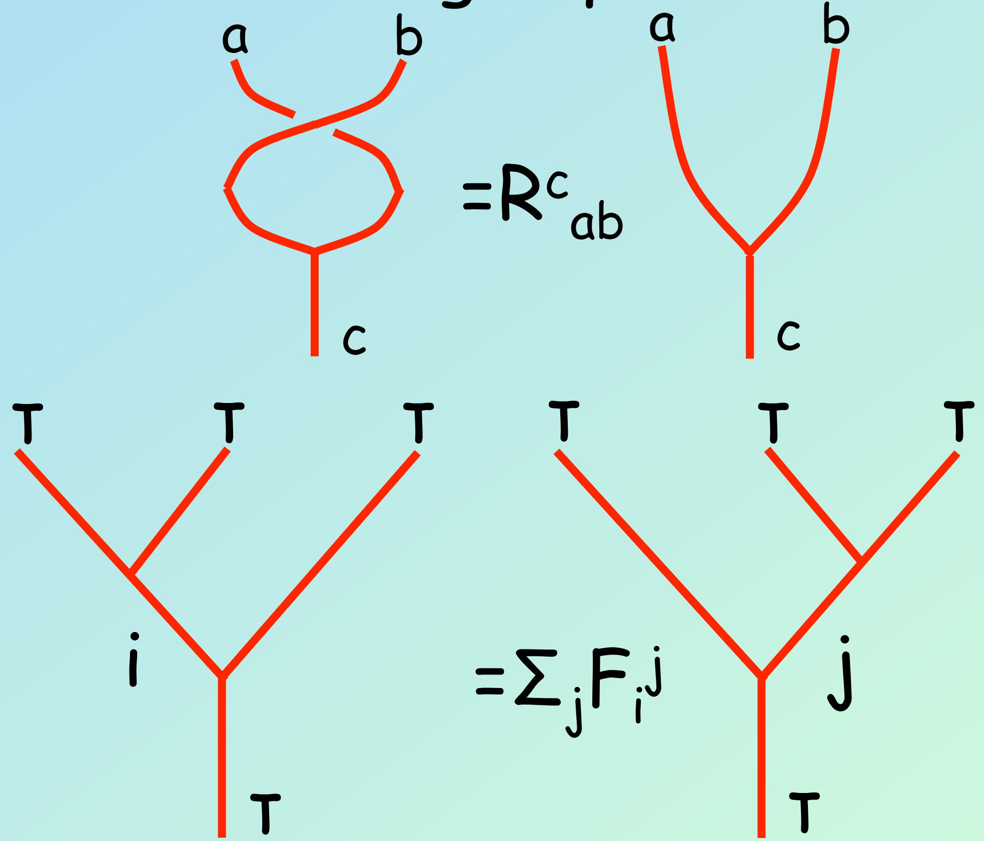
[Freedman, Larsen, Wang, CMP 228, 177 (2002)]

Fibonacci Anyons and QC

Qubit encoding:



Evolving a qubit:

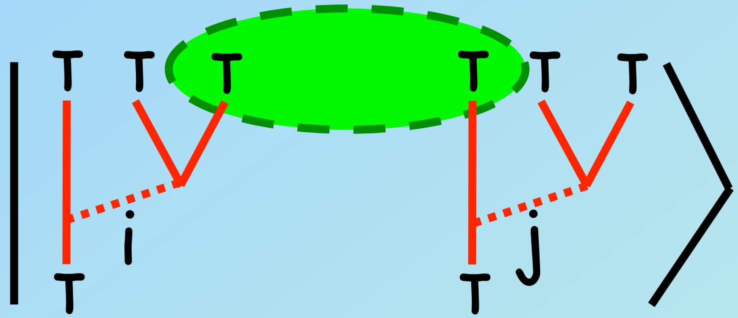


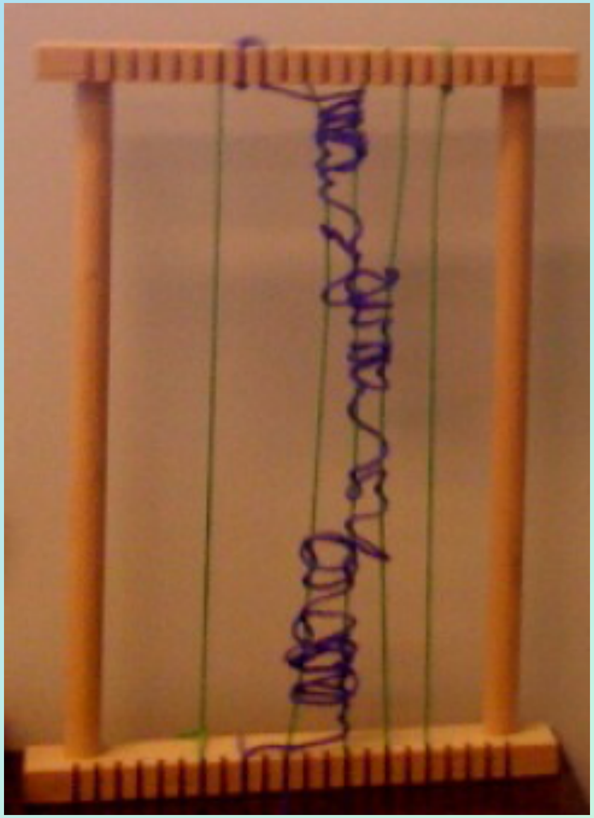
Unitaries B and F are dense in SU(2).

Extends to SU(d_n) when n anyons are employed.

Fibonacci Anyons and QC

Qubit encoding:





CNOT

Unitaries B and F are dense in SU(2).
Extends to $SU(d_n)$ when n anyons are employed.

Conclusions

- Topological Quantum Computation promises to **overcome** the problem of **decoherence** and errors in the most direct way.

- There is lots of work to be done to make anyons work for us.

- Is it worth it?

Aesthetics says YES!