# Multipartite Entanglement: Combinatorics, Topology and Astronomy

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**Sharif University**, **Teheran**, July 2, 2020

# Composed systems & entangled states

## bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

## Two–qubit system: $2 \times 2 = 4$

**Maximally entangled Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

## Schmidt decomposition & Entanglement measures

Any pure state from $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written by a **matrix** $G = U\Lambda V$

$|\psi\rangle = \sum_{ij} G_{ij}|i\rangle \otimes |j\rangle = \sum_i \sqrt{\lambda_i}|i'\rangle \otimes |i''\rangle$, where $|\psi|^2 = \mathrm{Tr}\, GG^\dagger = 1$.

The partial trace, $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi| = GG^\dagger$, has spectrum given by the **Schmidt vector** $\{\lambda_i\}$ = squared **singular values** of $G$, with $\sum_i \lambda_i = 1$.

Entanglement entropy of $|\psi\rangle$ is equal to **von Neumann entropy** of the reduced state $\sigma$

$$E(|\psi\rangle) := -\mathrm{Tr}\, \sigma \ln \sigma = S(\lambda).$$

# Maximally entangled bi–partite quantum states

## Bipartite systems $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}_d \otimes \mathcal{H}_d$

**generalized Bell state** (for two qu*d*its),

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

distinguished by the fact that all **singular values** are equal, $\lambda_i = 1/\sqrt{d}$,
hence the reduced state is **maximally mixed**,
$\rho_A = \mathrm{Tr}_B |\psi_d^+\rangle\langle\psi_d^+| = \mathbb{1}_d/d$.
This property holds for all locally equivalent states, $(U_A \otimes U_B)|\psi_d^+\rangle$.

**A)** State $|\psi\rangle$ is **maximally entangled** if $\rho_A = GG^\dagger = \mathbb{1}_d/d$,
which is the case if the **matrix** $U = \sqrt{d}G$ of size $d$ is **unitary**,
(and all its **singular values** are equal to 1),
e.g. for $G = H/2$ one has $|\Phi_{ent}\rangle = (|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$.
**B)** For a **bi–partite** state the **singular values** of $G$ characterize
**entanglement** of the state $|\psi\rangle = \sum_{i,j} G_{ij}|i,j\rangle$.

# Multi-partite pure quantum states

**What means:** **Multi-partite** ?

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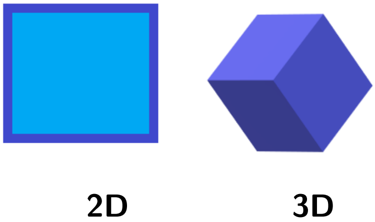
**Tres faciunt collegium**




**Multi** $= \quad N \geq 3 \quad ?$

# Multi-partite pure quantum states: $3 \gg 2$

States on $N$ parties are determined by a **tensor** with $N$ indices
e.g. for $N = 3$ : $|\Psi_{ABC}\rangle = \sum_{i,j,k} T_{i,j,k} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C$.

Mathematical problem: in general for a **tensor** $T_{ijk}$ there is no (unique) **Singular Value Decomposition** and it is not simple to find the **tensor rank** or **tensor norms** (nuclear, spectral) – see arXiv: **1912.06854**

**W. Bruzda, S. Friedland, K. Ż. (2019)**

*Tensor rank and entanglement of pure quantum states*

Open question: Which state of $N$ subsystems with $d$–levels each is the **most entangled** ?

**example for three qubits,** $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}_2^{\otimes 3}$

**GHZ** state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ has a similar property:
all three one-partite reductions are **maximally mixed**
$\rho_A = Tr_{BC}|GHZ\rangle\langle GHZ| = \mathbb{1}_2 = \rho_B = Tr_{AC}|GHZ\rangle\langle GHZ|$.
(what is **not** the case e.g. for $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$

# Genuinely multipartite entangled states

## $k$-uniform states of $N$ qu$d$its

**Definition**. State $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ is called ***k*-uniform**
if for all possible splittings of the system into $k$ and $N-k$ parts the reduced states are maximally mixed (**Scott 2001**),
(also called **MM**-states (maximally multipartite entangled)
**Facchi et al.** (2008,2010), **Arnaud & Cerf** (2012)

**Applications:** quantum error correction codes, teleportation, etc...

## Example: 1–uniform states of $N$ qu$d$its

**Observation.** A generalized, $N$–qu$d$it **GHZ** state,

$$|GHZ_N^d\rangle := \frac{1}{\sqrt{d}}\left[|1,1,...,1\rangle + |2,2,....,2\rangle + \cdots + |d,d,...,d\rangle\right]$$

is 1–**uniform** (but not 2–uniform!)

# Examples of $k$–uniform states

**Observation:** $k$–uniform states may exist if $N \geq 2k$ (**Scott 2001**) (traced out ancilla of size $(N-k)$ cannot be smaller than the principal $k$–partite system).

Hence there are no 2-uniform states of 3 **qubits**.

**However, there exist no 2-uniform state of 4 qubits either!**

**Higuchi & Sudbery** (2000) - **frustration** like in spin systems –
**Facchi, Florio, Marzolino, Parisi, Pascazio** (2010) –
it is not possible to satisfy simultaneously so many constraints...

**2-uniform state of 5 and 6 qubits**

$$|\Phi_5\rangle = |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle + \\ +|10000\rangle + |00101\rangle - |00011\rangle - |10110\rangle,$$

related to 5–qubit error correction code by **Laflamme et al.** (1996)

$$|\Phi_6\rangle = |111111\rangle + |101010\rangle + |001100\rangle + |011001\rangle + \\ +|110000\rangle + |100101\rangle + |000011\rangle + |010110\rangle.$$

$\Longrightarrow$ An introduction to "*Quantum Combinatorics*"

**A classical example:**

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an $4 \times 4$ array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

# Combinatorial Designs

$\Longrightarrow$ An introduction to *"Quantum Combinatorics"*

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b) - in every row and column there is only a **single** card of each rank

| | | | |
|---|---|---|---|
| $A\spadesuit$ | $K\diamondsuit$ | $Q\heartsuit$ | $J\clubsuit$ |
| $K\heartsuit$ | $A\clubsuit$ | $J\spadesuit$ | $Q\diamondsuit$ |
| $Q\clubsuit$ | $J\heartsuit$ | $A\diamondsuit$ | $K\spadesuit$ |
| $J\diamondsuit$ | $Q\spadesuit$ | $K\clubsuit$ | $A\heartsuit$ |

Two **mutually orthogonal Latin squares** of size $N = 4$

**Graeco–Latin** square !

# Mutually orthogonal Latin Squares (MOLS)

♣) $N = 2.$ There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)
♡) $N = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(N-1)$ MOLS.
♠) $N = 6.$ Only a **single** Latin Square exists (No OLS!).

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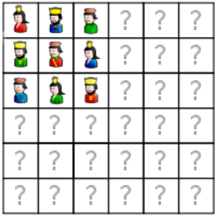
**Euler**'s problem: **36** officers of six different ranks from six different units come for a **military parade**. Arrange them in a square such that in each row / each column all uniforms are different.



**No solution exists** ! (conjectured by **Euler**), proof by:
**Gaston Terry** "Le Probléme de 36 Officiers". *Compte Rendu* **(1901)**.

# Absolutely maximally entangled state (AME)

**Homogeneous** systems (subsystems of the same kind)

**Definition.** A $k$–uniform state of $N$ qu*d*its is called **absolutely maximally entangled AME(N,d)** if $k = [N/2]$

Examples:

a) **Bell state** - 1-uniform state of 2 qubits = AME(2,2)

b) **GHZ state** - 1-uniform state of 3 qubits = AME(3,2)

x) **none** - no 2-uniform state of 4 qubits
**Higuchi & Sudbery** (2000)

c) 2-uniform state $|\Psi_3^4\rangle$ of 4 qutrits, AME(4,3)

d) 3-uniform state $|\Psi_4^6\rangle$ of 6 ququarts, AME(6,4)

e) no 3-**uniform** states of 7 qubits
**Huber, Gühne, Siewert** (2017)

# Higher dimensions: AME(4,3) state of four qutrits

From a **Greaco-Latin square** (= a pair of orthogonal **Latin squares**) of size $N = 3$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\alpha 0$ | $\beta 1$ | $\gamma 2$ | = | $A\spadesuit$ | $K\clubsuit$ | $Q\diamondsuit$ |
| $\gamma 1$ | $\alpha 2$ | $\beta 0$ | | $K\diamondsuit$ | $Q\spadesuit$ | $A\clubsuit$ |
| $\beta 2$ | $\gamma 0$ | $\alpha 1$ | | $Q\clubsuit$ | $A\diamondsuit$ | $K\spadesuit$ |

we get a 2–**uniform** state of **4 qutrits**:

$$
\begin{aligned}
|\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

Corresponding **Quantum Code**:

$$
\begin{aligned}
|0\rangle &\to |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\to |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\to |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$

# Why do we care about AME states?

Since they can be used for various purposes
(e.g. **Quantum codes**, **teleportation**,...)

Resources needed for **quantum teleportation**:

a) **2-qubit Bell state** allows one to teleport **1 qubit** from A to B

b) **2-qudit generalized Bell state** allows one to teleport **1 qudit**

c) **3-qubit GHZ state** allows one to teleport **1 qubit** between any users

d) **4-qutrit GHZ state** allows one to teleport **1 qutrit**
between any two out of four users

f) **4-qutrit state AME(4,3)** allows one to teleport **2 qutrits** between
**any** pair chosen from four users to the other pair!
- say from the pair (A & C) to (B & D)

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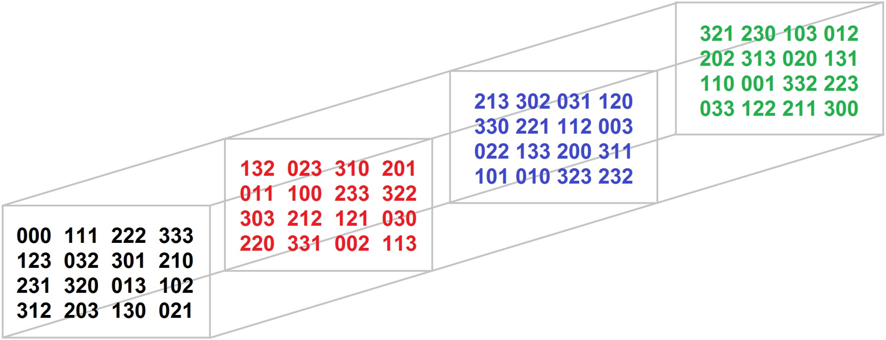
relations between **AME states** and **multiunitary matrices, perfect tensors and holographic codes**

# State AME(6,4) of six ququarts:

3–uniform state of **6 ququarts**: read from three **Mutually orthogonal Latin cubes**

$|\Psi_4^6\rangle =$

$$|000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle +$$
$$|012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle + |022013\rangle + |023102\rangle +$$
$$|030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle +$$
$$|102310\rangle + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle +$$
$$|120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle + |130220\rangle + |131331\rangle +$$
$$|132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle +$$
$$|210330\rangle + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle +$$
$$|222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle + |232323\rangle + |233232\rangle +$$
$$|300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle +$$
$$|312020\rangle + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle +$$
$$|330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle.$$




State $|\Psi_4^6\rangle$ of **six** **ququarts** can be generated by three mutually orthogonal **Latin cubes of order four!**

(three address quarts + three cube quarts = 6 quarts in $4^3 = 64$ terms)

# Absolutely maximally entangled state (AME) II

**Key issue** For what number $N$ of qu*d*its the state **AME(N,d)** exist?

How to construct them??

AME(5,2) [**five qubits**] and AME(6,2) [**six qubits**] do exist

but

they contain terms with negative signs $\Rightarrow$ cannot be obtained with Latin squares

new construction needed...

*"every good notion can be quantized"*

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The new notion of

**Quantum Latin Square** (QLS) by **Musto & Vicary** (2016)

(square array of $N^2$ quantum states from $\mathcal{H}_N$:

every column and every row forms a basis)

inspired us to introduce

**Mutually Orthogonal Quantum Latin Squares** (MOQLS)

# Superpositions, entangled states and "quantum designs"

## Quantum orthogonal Latin square

Example of order $N = 4$ by **Vicary, Musto (2016)**

$$\begin{vmatrix} |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |2\rangle & |1\rangle & |0\rangle \\ |\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\ |\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle \end{vmatrix}$$

where $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ denote **Bell states**, while
$|\xi_+\rangle = \frac{1}{\sqrt{5}}(i\,|0\rangle + 2\,|3\rangle)$ $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2\,|0\rangle + i\,|3\rangle)$ other **entangled** states.

Four states in each row & column form an **orthogonal basis** in $\mathcal{H}_4$

Standard **combinatorics**: **discrete** set of symbols, $1, 2, \ldots, N$,
$+$ **permutation** group

generalized ("Quantum") **combinatorics**: **continuous** family
of states $|\psi\rangle \in \mathcal{H}_N$ $+$ **unitary** group $U(N)$.

# Orthogonal Quantum Latin Squares

*"every good notion can be quantized"*

**Definition**. A table of $N^2$ bipartite states $|\phi_{i,j}\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$

$$QOLS = \begin{pmatrix} |\phi_{11}\rangle & |\phi_{12}\rangle & \dots & |\phi_{1N}\rangle \\ |\phi_{21}\rangle & |\phi_{22}\rangle & \dots & |\phi_{2N}\rangle \\ \dots & \dots & \dots & \dots \\ |\phi_{N1}\rangle & |\phi_{N2}\rangle & \dots & |\phi_{NN}\rangle \end{pmatrix}$$

forms a pair of two **Orthogonal Quantum Latin Squares** if:

**a)** all $N^2$ states are mutually orthogonal, $\langle\phi_{ij}|\phi_{kl}\rangle = \delta_{ik}\delta_{jl}$,

**b)** superpositions of all $N$ states in each row (each column) $\sum_{i=1}^{N}|\phi_{ij}\rangle$ and $\sum_{i=1}^{N}|\phi_{ji}\rangle$ are maximally entangled (=1 uniform) for $j = 1, \dots, N$.

Then the 4-partite state $|\Psi_4\rangle := \sum_{i=1}^{N}\sum_{j=1}^{N}|i,j\rangle \otimes |\phi_{ij}\rangle$ is 2–uniform, so it forms the state $|AME(4, N)\rangle$.

**Goyeneche, Raissi, Di Martino, K.Ż. Phys. Rev. A (2018)**

# Mutually Orthogonal Quantum Latin Cubes

*"every good notion can be quantized"*

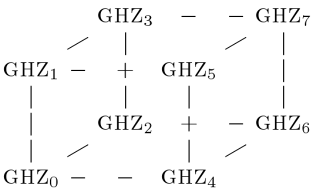
**Definition.** A cube of $N^3$ states $|\phi_{ijk}\rangle \in \mathcal{H}_N^{\otimes 3}$ forms a **Mutually Orthogonal Latin Cube** if the 6-party superposition

$$|\Psi_6\rangle := \sum_{i,j,k=1}^{N} |i,j,j\rangle \otimes |\phi_{ijk}\rangle$$ is 3–uniform

(so it forms the state $|AME(6,N)\rangle$).

**Example.** Cube of 8 states forming three-qubit GHZ basis:

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | $|\mathrm{GHZ}_0\rangle$ |
| 0 | 0 | 1 | $|\mathrm{GHZ}_1\rangle$ |
| 0 | 1 | 0 | $|\mathrm{GHZ}_2\rangle$ |
| 0 | 1 | 1 | $|\mathrm{GHZ}_3\rangle$ |
| 1 | 0 | 0 | $|\mathrm{GHZ}_4\rangle$ |
| 1 | 0 | 1 | $|\mathrm{GHZ}_5\rangle$ |
| 1 | 1 | 0 | $|\mathrm{GHZ}_6\rangle$ |
| 1 | 1 | 1 | $|\mathrm{GHZ}_7\rangle$ |



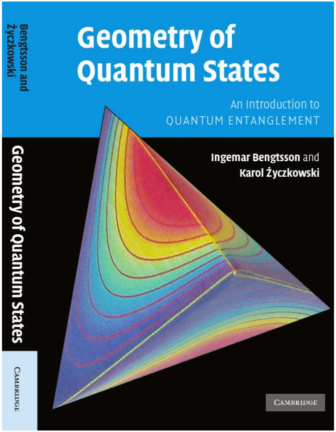

leads to the six-qubit AME state of **Borras**

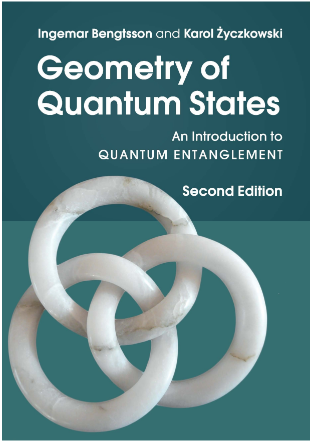
$$|AME(6,2)\rangle = \sum_{x=0}^{7} |x\rangle \otimes |GHZ_x\rangle.$$

(analogy to state $|\Psi(f)\rangle = \sum_x |x\rangle \otimes |f(x)\rangle$ used in the Shor algorithm!)

# Multipartite entanglement discussed in a book

published by Cambridge University Press in **2006**,



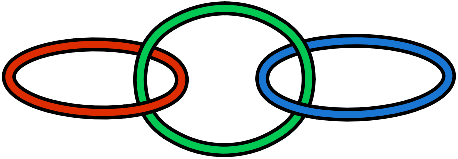



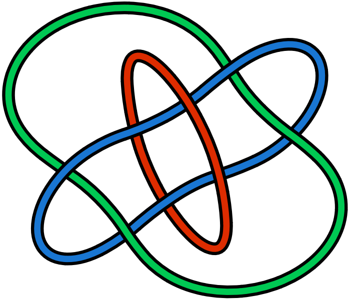

$|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$

**II edition (2017)** (with new chapters on multipartite entanglement & discrete structures in the Hilbert space),


**Literature suggested:** **Sznurkowe zwierzaki**

What 3-qubit **quantum state** can be associated with these links ?





# Topology: knots and links

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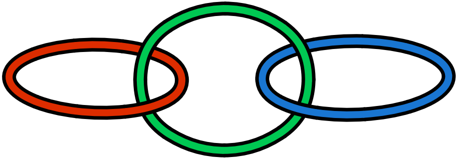
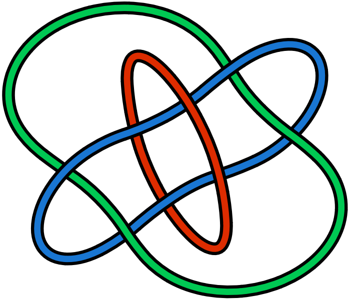




$P_3(a, b, c) = ab + bc$

if $b = 0$ then $P_3(a, b, c) = 0$

$P'_3(a, b, c) = abc$

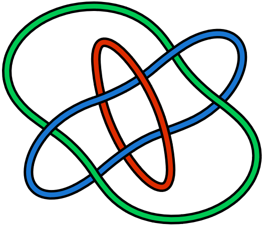
if $a = 0$ or $b = 0$ or $c = 0$

then $P'_3(a, b, c) = 0$

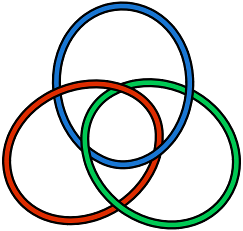
# Analogy: linked rings and quantum states

**Entangled** state of $n$ parties is visualized by a set of $n$ **linked** rings.

Interpretation of **cutting** (or neglecting) a ring $x$:
A) **Aravind (1997)** - after measurement of particle $x$ the remaining $n-1$ parties are in a **separable** state – **basis dependent**
B) **Sugita (2006)** - after partial trace over particle $x$ the remaining $n-1$ subsystems are in a **separable** state – **basis independent**



$P'_3(a,b,c) = abc$
$|GHZ_3\rangle = |000\rangle + |111\rangle$



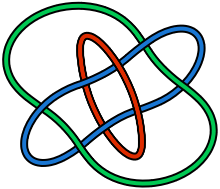
$P''_3(a,b,c) = ab + bc + ac$
$|W_3\rangle = |001\rangle + |010\rangle + |100\rangle$

# $m$-resistant links & $m$-resistant states

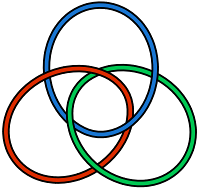
**Definition A.** A link of $n$ **rings** is called ***m*–resistant** if cutting *any* $m$ rings the remaining $n - m$ rings are **connected**, while cutting *any* further ring **disconnects** the link.

**Definition B.** A quantum state of $n$ **subsystems** is called ***m*–resistant** if after tracing away *any* $m$ subsystems the remaining $n - m$ parties remain **entangled**, while removing any other party makes the state **separable**.

**Examples:**



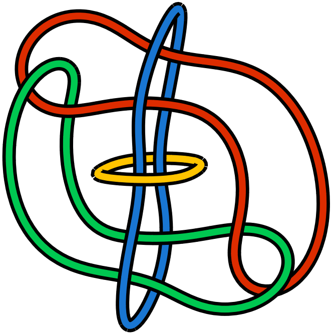
$|GHZ_3\rangle = |000\rangle + |111\rangle$
0–**resistant state**



$|W_3\rangle = |001\rangle + |010\rangle + |100\rangle$
1–**resistant state**

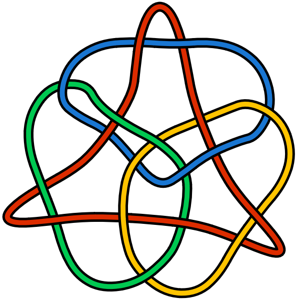
# Four Links & four-qubit states

What 4-qubit **quantum state** can be associated with these links ?



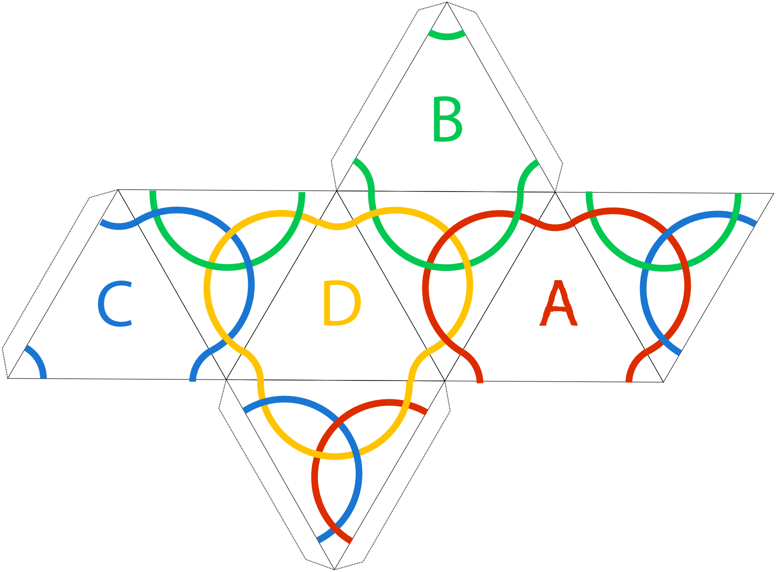
$P_4(a, b, c, d) = abcd$

0**–resistant link**



$P'_4(a, b, c, d) =$
$= abc + abd + acd + bcd$

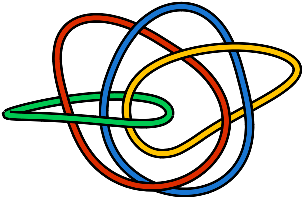
1**–resistant link**




**Four Borromean rings at an octahedron: 1-resistant link**

# $m$-resistant links & $m$-resistant states

**Statement A.** For any natural $n$ and $m < n - 1$ there exist an ***m*–resistant** link of $n$ **rings**.



$$P_4''(a, b, c, d) = ab + cd + ac + bd + ad + cb$$

2–**resistant link**

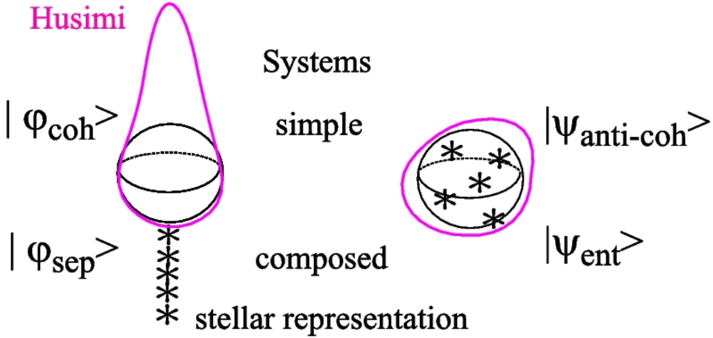
**Conjecture B.** For any natural $n$ and $m < n - 1$ there exist an ***m*–resistant** state of $n$ **subsystems**.
(in some cases general the states has to be mixed, and the local dimension $d > 2$.)

# Multipartite quantum states & Astronomy

# Stellar representation of $n$–qubit symmetric states

**Majorana** (stellar) representation of a permutation symmetric 2–qubit state: $|\Psi_2\rangle = \mathcal{N}[|\alpha,\beta\rangle + |\beta\alpha\rangle]$
consists of two points $\alpha$ and $\beta$ at the sphere ($=$ 2 stars at the sky).

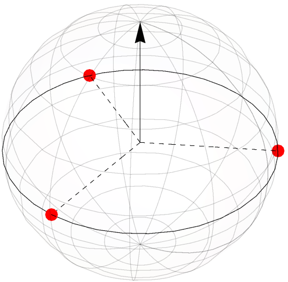
Any **constellation** of $n$ stars represents a symmetric state of $n$ qubits
$|\Psi_n\rangle = \mathcal{N} \ \sum_\sigma |\alpha_1\rangle_{i_1} \otimes \cdots \otimes |\alpha_n\rangle_{i_n}$,
where the sum goes over all $n!$ permutations $\sigma$.

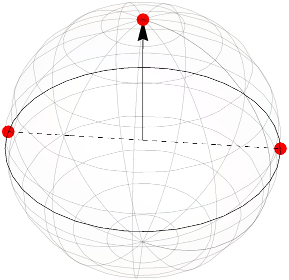


# $m$-resistant 3-qubit states & 3-star constellations

**Examples**

1. ***m*–resistant** pure states of $n = 3$ qubits
   represented by constellations of **three** stars, * * *, at the sky





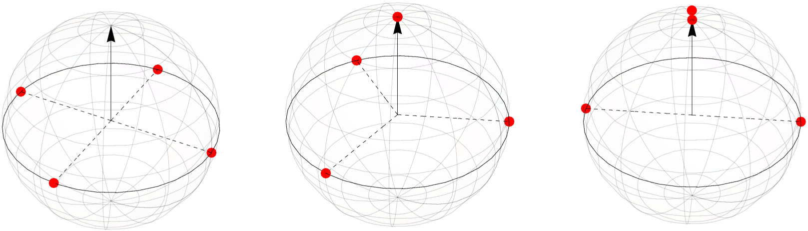
**0–resistant state**

$|GHZ_3\rangle = |000\rangle + |111\rangle$

**1–resistant state**

$3|000\rangle + |011\rangle + |101\rangle + |110\rangle$

2. $m$**–resistant** pure states of $n = 4$ qubits
   represented by constellations of **four** stars, * * * *, at the sky



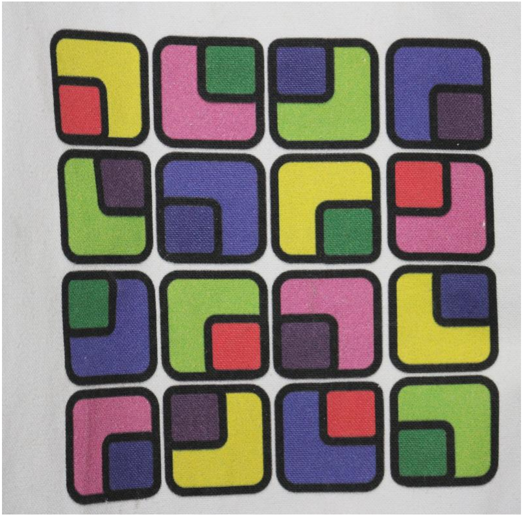
0–**resistant** 1–**resistant** 2–**resistant states**

$|GHZ_4\rangle = |0000\rangle + |1111\rangle$

3. Asymptotic case: generic state of $n$ subsystems with $d$-level each
   is typically $m$**–resistant** with $m \approx 3n/5$

**Quinta, André, Burhardt, K.Ż.** Phys. Rev. **A100, (2019)**

# A quick quiz



What **quantum state** can be associated with this design ?

# Hints

| | | | |
|---|---|---|---|
| $A\spadesuit$ | $K\diamondsuit$ | $Q\heartsuit$ | $J\clubsuit$ |
| $K\heartsuit$ | $A\clubsuit$ | $J\spadesuit$ | $Q\diamondsuit$ |
| $Q\clubsuit$ | $J\heartsuit$ | $A\diamondsuit$ | $K\spadesuit$ |
| $J\diamondsuit$ | $Q\spadesuit$ | $K\clubsuit$ | $A\heartsuit$ |

Two **mutually orthogonal** **Latin squares** of size $N = 4$

# Hints

| | | | |
|---|---|---|---|
| $A\spadesuit$ | $K\diamondsuit$ | $Q\heartsuit$ | $J\clubsuit$ |
| $K\heartsuit$ | $A\clubsuit$ | $J\spadesuit$ | $Q\diamondsuit$ |
| $Q\clubsuit$ | $J\heartsuit$ | $A\diamondsuit$ | $K\spadesuit$ |
| $J\diamondsuit$ | $Q\spadesuit$ | $K\clubsuit$ | $A\heartsuit$ |

Two **mutually orthogonal Latin squares** of size $N = 4$

| | | | |
|---|---|---|---|
| $A\spadesuit$ | $K\diamondsuit$ | $Q\heartsuit$ | $J\clubsuit$ |
| $K\heartsuit$ | $A\clubsuit$ | $J\spadesuit$ | $Q\diamondsuit$ |
| $Q\clubsuit$ | $J\heartsuit$ | $A\diamondsuit$ | $K\spadesuit$ |
| $J\diamondsuit$ | $Q\spadesuit$ | $K\clubsuit$ | $A\heartsuit$ |

**Three mutually orthogonal Latin squares** of size $N = 4$

# The answer

Bag shows **three mutually orthogonal Latin squares** of size $N = 4$ with three attributes $A, B, C$ of each of $4^2 = 16$ squares.
Appending two indices, $i, j = 0, 1, 2, 3$ we obtain a $16 \times 5$ table,
$A_{00}, B_{00}, C_{00}, 0, 0$
$A_{01}, B_{01}, C_{01}, 0, 1$
........................
$A_{33}, B_{33}, C_{33}, 3, 3.$
It leads to the 2–**uniform** state of **5 ququarts**,

$$
\begin{aligned}
|\Psi_4^5\rangle = \quad & |00000\rangle + |12301\rangle + |23102\rangle + |31203\rangle + \\
& |13210\rangle + |01111\rangle + |30312\rangle + |22013\rangle + \\
& |21320\rangle + |33021\rangle + |02222\rangle + |10123\rangle + \\
& |32130\rangle + |20231\rangle + |11032\rangle + |03333\rangle
\end{aligned}
$$

related to the **Reed–Solomon code** of length 5.

# Concluding Remarks

1. **Strongly entangled** multipartie quantum states can be useful for quantum error correction codes, multiuser quantum communication and other protocols
2. In some cases it is unknown, whether there exists an absolutely maximally entangled state **(AME)** of $n$ qu$d$its.
   **Open** issue: 4 subsystems with $d = 6$ levels each related to the problem of 36 **entangled officers** of Euler. Recent numerical results suggest that such the corresponding state $AME(4, 6)$ **does not** exist.
   **Bruzda, Rajchel, Lakshminarayan, K.Ż.** (2020), *to appear*

To construct **strongly entangled** states of several qudits we advocate:

1. (a) **combinatorial** techniques (**quantum** orthogonal Latin squares)
2. (b) **topological** techniques ($m$–resistant links and states)
3. (c) application of **stellar representation**
4. construction of $k$-uniform mixed states: **Kłobus, Burchardt, Kołodziejski, Pandit, Vertesi, K. Ż. and Laskowski**, PRA (2019).