EBERHARD KARLS UNIVERSITÄT TÜBINGEN
FACULTY OF SCIENCE
Institute of Theoretical Physics

Probing the quantumness of states and channels with truncated moment sequences

DANIEL BRAUN

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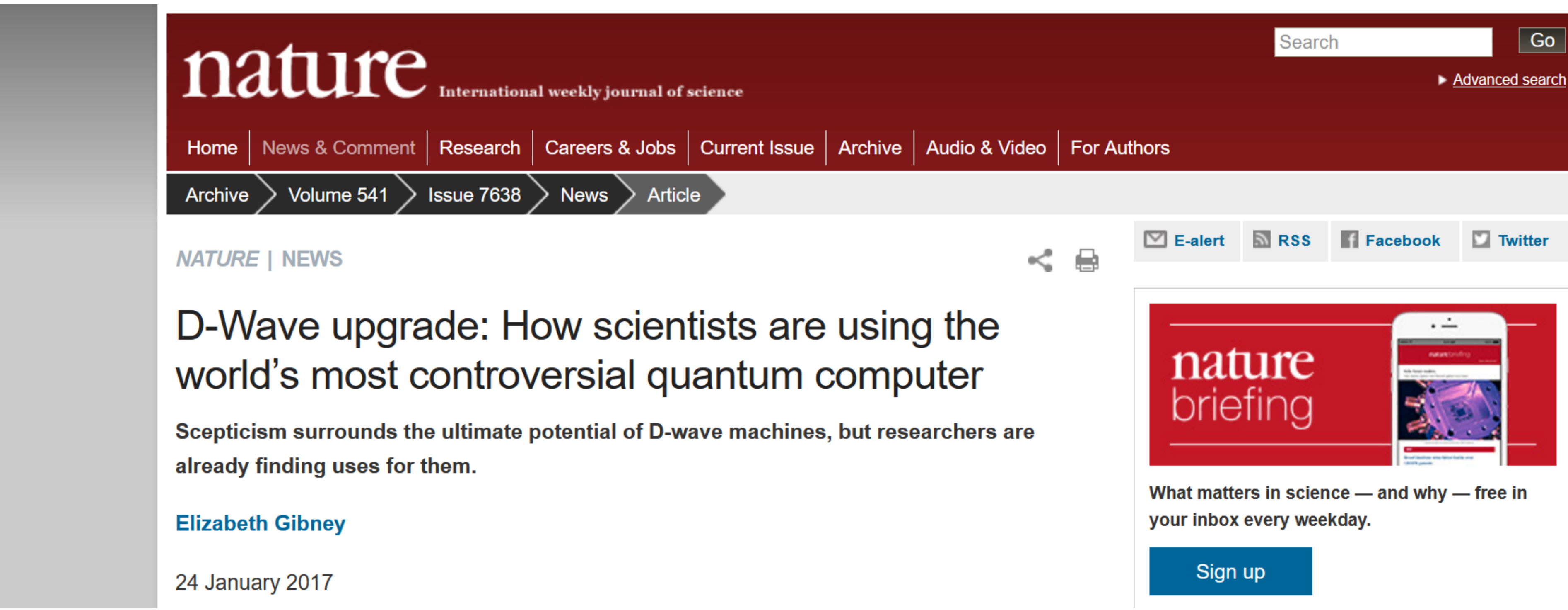
Motivation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Scepticism surrounds the ultimate potential of D-wave machines, but researchers are already finding uses for them.

Elizabeth Gibney

24 January 2017

nature briefing

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Quantum – or not ?

Quantum – or not?

- How to define „quantumness“ ?
  - Quantum interference/superpositions of „classical“ states
  - Quantum noise
  - Entanglement

- How to detect „quantumness“ efficiently?

- Of states and channels…

I. Introduction

Entanglement problem

- Definition: state ρ A|B separable:

$$\rho = \sum_i p_i \rho_i^{(A)} \otimes \rho_i^{(B)}$$

$$p_i \geq 0, \ \sum_i p_i = 1$$

If not, ρ A|B „entangled“

Given ρ, decide whether ρ is entangled or separable.

- Easy if ρ is pure: von Neumann entropy of reduced state >0 $\Leftrightarrow$ ρ entangled
- ρ mixed: necessary and sufficient criteria for separability
  - PPT (positive partial transpose); in general: positive but not completely positive maps [Peres ‘96, Horodeckis ‘96]
  - Entanglement witnesses (Positive operators on separable states)
  - Hierachy of semi-definite programs based on symmetric, flat PPT extensions [Doherty, Parillo, Spedallieri PRL, PRA 2002-2005]
  - NP hard in general [Gurvits 2003]

Bloch-tensor representation

- Qudit: q-system with d-dimensional Hilbert space
- spin-j state = symmetric state of N=2j qubits, with d=2j+1=N+1
- Representable with Bloch-tensor!

Symmetric states of N qubits – spin-j

Single qubit – spin-1/2

Bloch picture:

$$\rho = \frac{1}{2}(\mathbf{1}_2 + \boldsymbol{n} \cdot \boldsymbol{\sigma}) = \frac{1}{2} X_\mu S_\mu \qquad \mu \in \{0, 1, 2, 3\}$$

$S_i = \sigma_i$ Pauli matrices

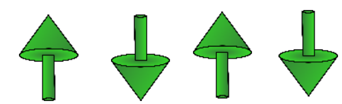
$\boldsymbol{n} \in \mathbb{R}^3$ Bloch vector

$$X_\mu = (1, \boldsymbol{n})$$

Symmetric states of N qubits – spin-j

- Basis of pure symmetric states spanned by „Dicke states“

$$|D_N^{(k)}\rangle = \mathcal{N} \sum_{\pi} |\underbrace{0\ldots0}_{N-k}\underbrace{1\ldots1}_{k}\rangle, \quad k = 0, \ldots, N,$$



→ N+1 dim subspace $\mathbb{C}^{N+1} \subseteq \mathbb{C}^{(2^N)}$

- Projector onto symmetric states: $P_S = \sum_{k=0}^{N} |D_N^{(k)}\rangle\langle D_N^{(k)}|$

- Mixed state „basis“: Weinberg matrices (tight frame) S. Weinberg, PR 1964

$$S_{\mu_1\ldots\mu_N} = P_S \left(\sigma_{\mu_1} \otimes \sigma_{\mu_2} \cdots \otimes \sigma_{\mu_N}\right) P_S^{\dagger} \in \mathcal{M}_{N+1}(\mathbb{C}), \quad 0 \leq \mu_i \leq 3,$$

O. Giraud, DB, D. Baguette, T. Bastin, and J. Martin, PRL 2015

Symmetric states of N qubits – spin-j

$$\rho = \frac{1}{2^N} X_{\mu_1\mu_2\ldots\mu_N} S_{\mu_1\mu_2\ldots\mu_N}$$

tight frame property

$$X_{\mu_1\mu_2\ldots\mu_N} = \mathrm{tr}(\rho\, S_{\mu_1\mu_2\ldots\mu_N})$$

Bloch tensor: real symmetric tensor of rank *N*, dimension 4

- Reduced density matrix of *N-k* qubits $\quad \mathrm{tr}_k\rho \to X_{\mu_1\ldots\mu_{N-k}0\ldots0}$

  $\mathrm{tr}\rho = 1 \to X_{0\ldots0} = 1$

- Transforms by rotation (per index) under SU(2) trafos
- Unique decomposition if imposing „relativistic tracelessness“

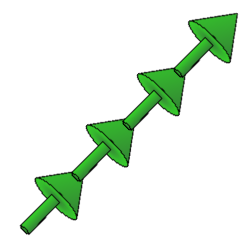
$$g_{\mu_1\mu_2} X_{\mu_1\mu_2\ldots\mu_N} = 0,\ g = \mathrm{diag}(-,+,+,+)$$

Entanglement of symmetric qubit states

Pure separable symmetric state => ρ fully separable (any bipartition)

- SU(2) „spin coherent state“ of a spin-j with j=N/2
- minimal quantum fluctuations
- „most classical“ pure spin-j state for a physical spin!

$$X_{\mu_1 \ldots \mu_N} = n_{\mu_1} n_{\mu_2} \ldots n_{\mu_N}$$



$$|\alpha\rangle = |\theta, \phi\rangle$$

O.Giraud, P.A. Braun, DB, PRA 2010

Separable spin-j states

Symmetric separable state of N qubits = „classical“ spin-state of spin j=N/2

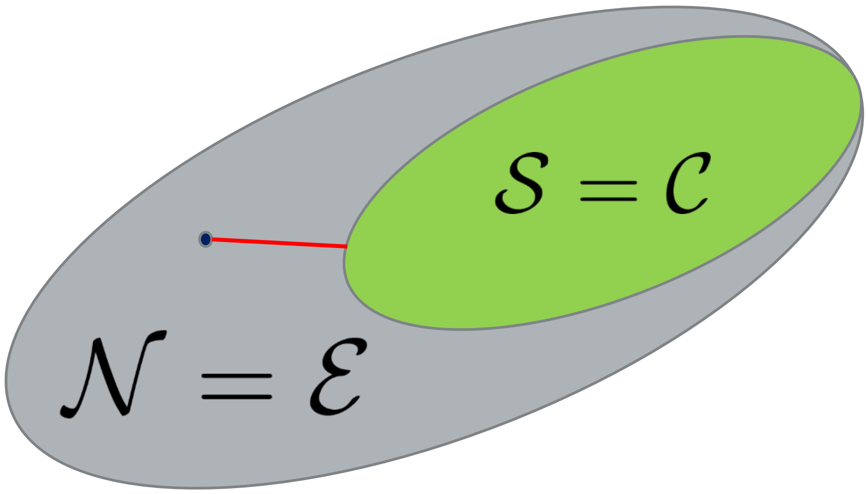
- Classical mixture of „most classical“ pure spin-j states

$$X_{\mu_1\ldots\mu_N} = \sum_i p_i n^{(i)}_{\mu_1} n^{(i)}_{\mu_2} \ldots n^{(i)}_{\mu_N} \quad \Leftrightarrow \text{ separability}$$

- Convex set

- Quantumness

$$Q(\rho) \equiv \min_{\rho_c \in \mathcal{C}} ||\rho - \rho_c||$$



O.Giraud, P.A. Braun, DB, NJP 2010

II. Truncated moment sequences

Truncated moment problem

Given:

N.I. Akhiezer, The Classical Moment Problem, (1965); J. Nie, Found. Comput. Math. (2014)

- a truncated moment sequence (tms) of degree d, $y = (y_\alpha)_{|\alpha| \leqslant d}$,

  $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \sum_i \alpha_i$, $y_\alpha \in \mathbb{R}$

- a semi-algebraic set $K = \{\mathbf{x} \in \mathbb{R}^n | g_1(\mathbf{x}) \geqslant 0, \cdots, g_m(\mathbf{x}) \geqslant 0\}$,

  $g_i(\mathbf{x})$ multivariate polynomials in the variables $x_1, \ldots, x_n$

Does there exist a positive measure $d\mu$ on K such that $\forall y_\alpha$ with $|\alpha| \leqslant d$

$$y_\alpha = \int_K x^\alpha d\mu(\mathbf{x}) \,,$$

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} ?$$

If so, $d\mu(\mathbf{x}) = \sum_{j=1}^r w_j \delta\left(\mathbf{x} - \mathbf{y}^{(j)}\right)$ with some finite r and $w_j > 0$,

a "finite atomic representing measure".

TMS versus entanglement problem

ρ symmetric separable state of N qubits $\iff \exists\, p_j \geq 0,\, n^{(i)}$ such that

$$X_{\mu_1\mu_2\ldots\mu_N} = \sum_j p_j n^{(j)}_{\mu_1} \cdots n^{(j)}_{\mu_N},$$

$n^{(j)}_0 = 1$ and each Bloch vector $\mathbf{n}^{(j)}$ normalized to 1.

$$\iff y_\alpha = X_{\mu_1\mu_2\ldots\mu_N} = \int_K x_{\mu_1} x_{\mu_2} \cdots x_{\mu_N} d\mu(\mathbf{x}) = \int_K x^\alpha \, d\mu(\mathbf{x}),$$

with $K = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ the unit sphere,

$x_0 = 1$, and $d\mu$ a positive measure on K.

Notation: $\alpha = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_i \in \mathbb{N}_0$ $\qquad x_{\mu_1} x_{\mu_2} \cdots x_{\mu_N} = x^\alpha$

$y_\alpha = X_{\mu_1\mu_2\ldots\mu_N}$ $\quad$ e.g. for $N = 6$, $y_{(2,1,0)} = X_{000112}$

TMS machinery

- Moment matrix of order *k*:

$$M_k(y)_{\alpha\beta} = y_{\alpha+\beta}, \qquad |\alpha|, |\beta| \leqslant k$$

  - for the tms to have a solution, M_k must be positive semi-definite

$$(M_{2j})_{\boldsymbol{\alpha\beta}} = X_{\underbrace{\mu_1 \ldots \mu_j}_{\to\alpha} \underbrace{\nu_1 \ldots \nu_j}_{\to\beta}}$$

ρ^{T_A} and M_{2j} are similiar, i.e.

$$\exists R \text{ unitary and } \lambda > 0 \mid R^\dagger \rho^{T_A} R = \lambda T$$

F. Bohnet-Waldraff, DB, O.Giraud, PRA '16

⇒recover immediately PPT criterion!

- Shifted tms:

g a polynomial of degree $\deg(g) \geq 1$: $g(\boldsymbol{x}) = \sum_\gamma g_\gamma x^\gamma$

$$(g \star y)_\alpha = \sum_{|\gamma| \leqslant deg(g)} g_\gamma y_{\alpha+\gamma}, \quad |\alpha| \leqslant d - \deg(g)$$

TMS machinery

- Localizing matrix of order *k*: $d_g = \lceil deg(g)/2 \rceil$

$$M_{k-d_g}(g \star y)_{\alpha\beta} = (g \star y)_{\alpha+\beta} = \sum_{|\gamma| \leqslant deg(g)} g_\gamma y_{\alpha+\beta+\gamma}, \quad |\alpha|, |\beta| \leqslant k - d_g$$

  - with $g > 0$, also the localizing matrices must be positive

TMS machinery

- Localizing matrix of order *k*:

$$M_{k-d_g}(g \star y)_{\alpha\beta} = (g \star y)_{\alpha+\beta} = \sum_{|\gamma| \leqslant deg(g)} g_\gamma y_{\alpha+\beta+\gamma}, \quad |\alpha|, |\beta| \leqslant k - d_g$$

  - with *g* > 0, also the localizing matrices must be positive

- Extension of a tms:

$$\text{a tms } z \text{ of degree } 2k > d, \text{ such that } z_\alpha = y_\alpha \ \forall \ |\alpha| \leqslant d$$

TMS machinery

- Localizing matrix of order *k*:

$$M_{k-d_g}(g \star y)_{\alpha\beta} = (g \star y)_{\alpha+\beta} = \sum_{|\gamma| \leqslant deg(g)} g_\gamma y_{\alpha+\beta+\gamma}, \quad |\alpha|, |\beta| \leqslant k - d_g$$

  - with *g* > 0, also the localizing matrices must be positive

- Extension of a tms:

$$\text{a tms } z \text{ of degree } 2k > d, \text{ such that } z_\alpha = y_\alpha \ \forall\ |\alpha| \leqslant d$$

- Flat extension:

$$\text{rank} M_k(z) = \text{rank} M_{k-d_0}(z)$$

$$d_0 = \max_{1 \leq i \leq m}\{1, \lceil \deg(g_i)/2 \rceil\}$$

$$\text{here: } m = 2,\ g_1 = x_1^2 + x_2^2 + x_3^2 - 1, g_2 = -g_1, d_0 = 1$$

Solution of tms and entanglement problem

- Theorem: [Existence of solution of tms problem] Curto and Fialkow (2005)

A tms $(y_\alpha)_{|\alpha|\leqslant d}$ admits a representing measure supported by K iff there exists a flat extension $(z_\beta)_{|\beta|\leqslant 2k}$ with $2k > d$ such that $M_k(z) \geq 0$, $M_{k-d_{g_i}}(g_i \star z) \geqslant 0$ for $i = 1, \ldots, m$.

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- Theorem: [Separability of a general symmetric multi-partite state]

A state ρ is separable iff $X_{\mu_1\mu_2\ldots\mu_d}$ are mapped to a tms $(y_\alpha)_{\alpha\in\mathcal{A}}$ such that there exists a flat extension $(z_\beta)_{|\beta|\leqslant 2k}$ with $2k > d$, $M_k(z) \geqslant 0$, and $M_{k-d_{g_i}}(g_i \star z) \geqslant 0$ for $i = 1, \ldots, m$.

Bohnet-Waldraff, DB, Giraud, PRA 2017

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J. Nie, Found. Comput. Math. '14

- Theorem: [Separability of a general symmetric multi-partite state]

AK-tms

A state ρ is separable iff $X_{\mu_1\mu_2\ldots\mu_d}$ are mapped to a tms $(y_\alpha)_{\alpha\in\mathcal{A}}$ such that there exists a flat extension $(z_\beta)_{|\beta|\leqslant 2k}$ with $2k > d$, $M_k(z) \geqslant 0$, and $M_{k-d_{g_i}}(g_i \star z) \geqslant 0$ for $i = 1, \ldots, m$.

Bohnet-Waldraff, DB, Giraud, PRA 2017

Numerical solution by semi-definite program

Helton & Nie, Found. Comput. Math. (2012)

$$\min_z \sum_{\alpha,|\alpha|\leqslant k_0} R_\alpha z_\alpha \quad \text{such that} \tag{1}$$

$$M_k(z) \geqslant 0 \tag{2}$$

$$M_{k-d_i}(g_i \star z) \geqslant 0 \quad \text{for } i = 1,\ldots,m \tag{3}$$

$$z_\alpha = y_\alpha \text{ for } |\alpha| \leqslant d. \tag{4}$$

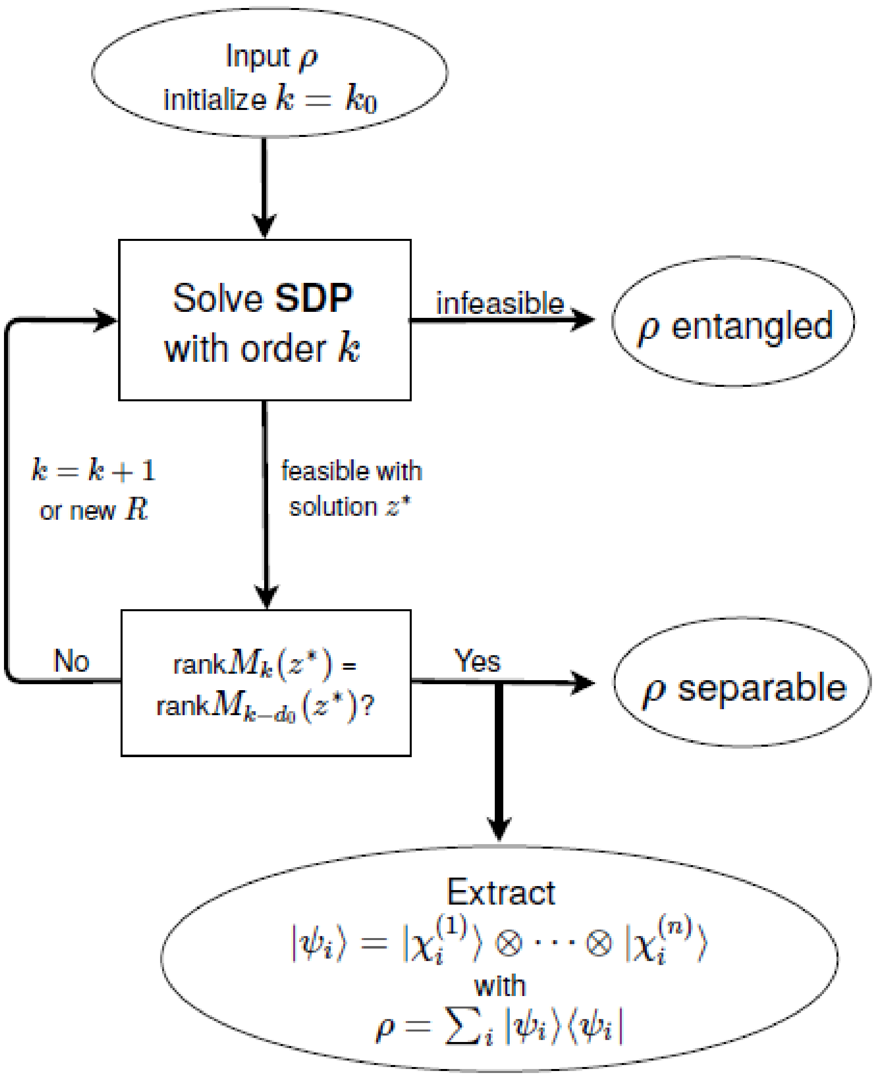
$R(\mathbf{x}) = \sum_\alpha R_\alpha x^\alpha$ taken as a random sum-of-squares polynomial of degree $2k_0$.

Kept the same when extension order increased.

$k_0 = \lfloor d/2 \rfloor + 1$

Numerical solution by semi-definite program

Helton & Nie, Found. Comput. Math. (2012)



- Implemented with “gloptypoly 3” in Matlab

  Henrion, Lasserre, Löfberg

- Flatness-check of extension not implementable by semi-definite program, otherwise N=NP

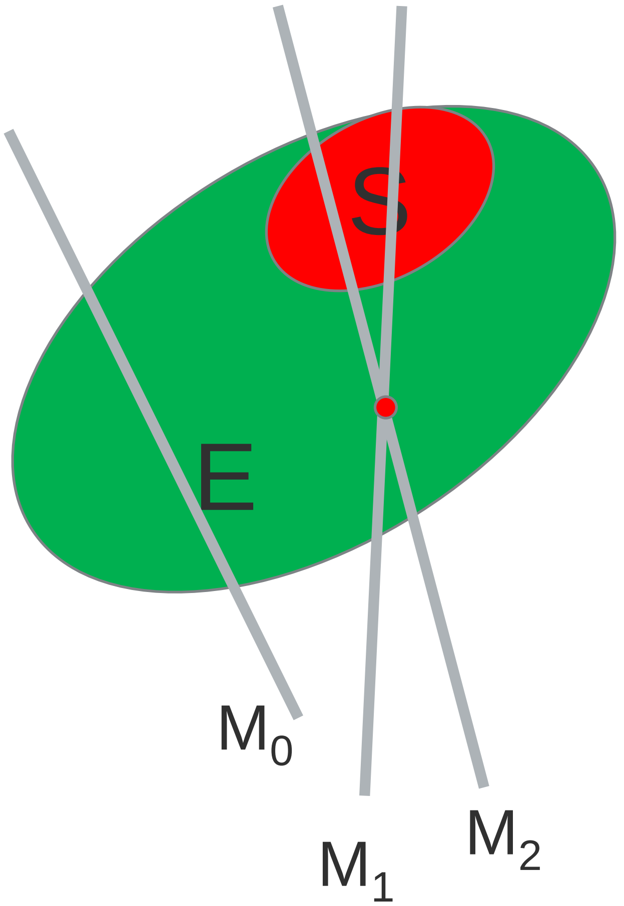
Performance and results

- Typical runtimes in seconds on a desktop computer:

States $\backslash N$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
ρ_{ent}	0.2	0.2	0.4	0.6	1.0	2.1	5.2	11.6	26.8	54.6	170.5
ρ_{sep}	0.7	0.4	0.6	1.0	2.0	4.2	10.2	20.8	66.9	94.5	716.3

  - 100 random *N*-qubit symmetric states
  - First row: random states drawn from the Haar measure (usually entangled, and typically detected by the condition $M_k(y) \not\geq 0$
  - Second row: random separable states from randomly mixing random pure separable states
  - For symmetric states, significantly outperforms current state of the art QETLAB toolbox

III. Fast entanglement detection



- Measurement (expectation value of observable) defines hyperplane
- Smallest set of measurements to guarantee intersection of hyperplanes in E?
- Most efficient sequence of measurements?

- AK-TMS approach ideally suited!

Smallest set of measurements

- Symmetric states of 2 qubits
- Allow Pauli measurements

$$\mathcal{M} = \{1, x, y, z, xx, xy, xz, yy, yz, zz\}$$

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Smallest set of measurements

- Symmetric states of 2 qubits
- Allow Pauli measurements

$$\mathcal{M} = \{\cancel{1}, x, y, z, xx, xy, xz, yy, yz, \cancel{zz}\}$$

trivial

redundant:

$$\sum_i X_{ii} = X_{00} = 1$$

Smallest set of measurements

$$\mathcal{M} = \{x, y, z, xx, xy, xz, yy, yz\}$$

- Freedom of choice of coordinate axes (irrelevant for statistics)

- Subsets of *k* elements that are non-equivalent under permutation of axes:

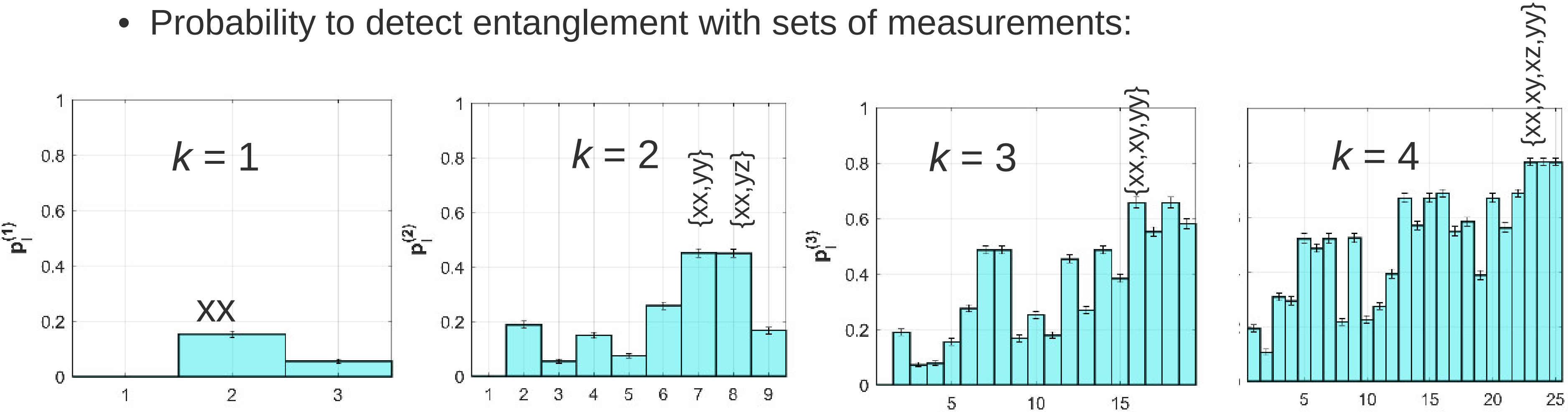
$k = 1$	
1	x
2	xx
3	xy

$k = 2$	
1	$\{x, y\}$
2	$\{x, xx\}$
3	$\{x, xy\}$
4	$\{x, yy\}$
5	$\{x, yz\}$
6	$\{xx, xy\}$
7	$\{xx, yy\}$
8	$\{xx, yz\}$
9	$\{xy, xz\}$

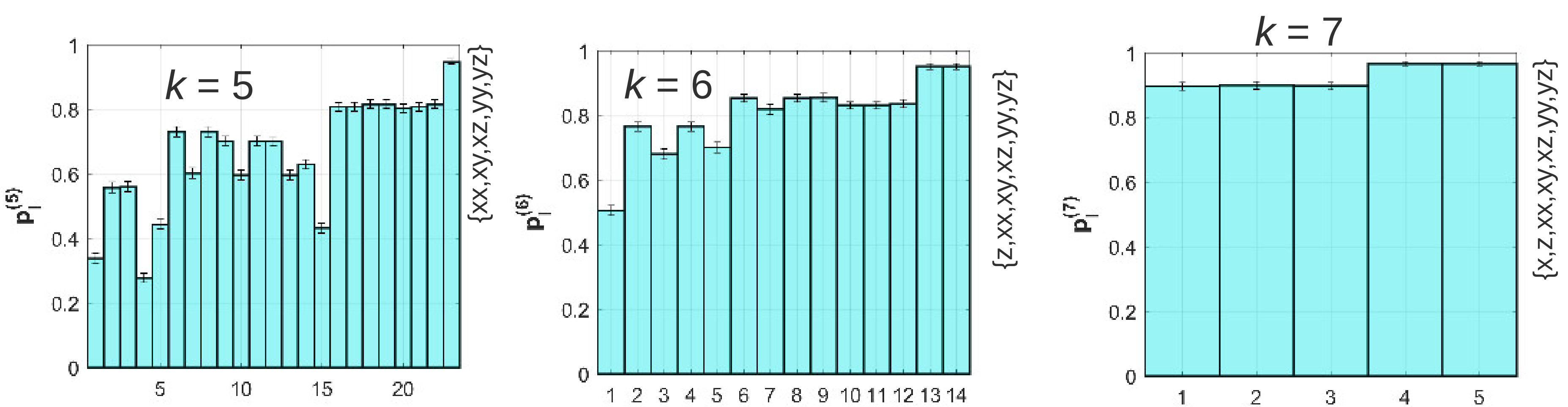
k =3…

Probability to detect entanglement

- 50000 randomly drawn states (Hilbert Schmidt ensemble), removed separable ones
- Probability to detect entanglement with sets of measurements:



Error bars = min/max from 1000 different sub-samples of size 40 000



N. Milazzo, DB, O. Giraud, PRA 2019

Optimal measurement sequence

- Detect entanglement as quickly as possible
- Optimal sequence (path γ) of measurements?

=> From sets to tupels. E.g. *k*=2:

$$\mathcal{M} = \{x, xx\} \rightarrow (x, xx), (xx, x)$$

- Always start with xx: success with *p*=0.18 after first measurement.
- Probability to stop after exactly *k* measurements:

$$r^{(k)}(\gamma) = p^{(k)}(\gamma) - p^{(k-1)}(\gamma)$$

with $$p^{(k)}(\gamma) \equiv p(E, \{M_1, \ldots, M_k\})$$

Probability to detect entanglement with the *k* measurements of path γ

Path length statistics

- Average (over states) path length:

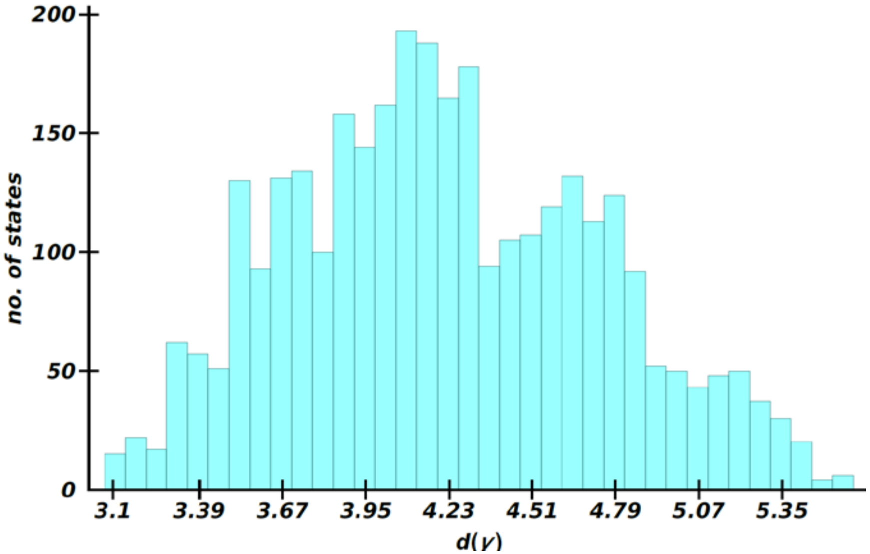
$$d(\gamma) = \sum_{k=1}^{8} k r^{(k)}(\gamma)$$

- Extremal paths:

$$\gamma_{\text{best}} = \arg\min_{\gamma \in S} d(\gamma) \simeq 3.07$$

$$\gamma_{\text{worst}} = \arg\max_{\gamma \in S} d(\gamma) \simeq 5.61$$

Distribution of average path lengths over set *S* of all 3228 inequivalent paths



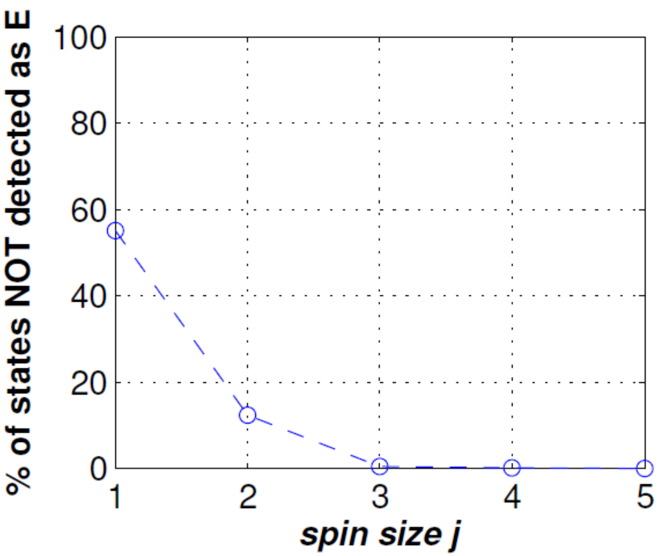
$$\gamma_{\text{best}} = (xx, yy, xz, yz, xy, x, y, z)$$

Measure in this order and need, on the average, only 3.07 measurements to certify entanglement.
Very incomplete information on state!

N. Milazzo, DB, O. Giraud, PRA 2019

Larger numbers of qubits, symmetric case

- Cardinality of sets of measurements grows rapidly with spin-j
- Focus on „diagonal“ observables: e.g. xx, xxyy, xxxxzz etc.
- State separable, integer *j* => these must be all positive
- Very efficient on average:



New type of sets of entanglement witnesses:

$$\binom{j+3}{3}$$ many observables

E.g. j=6 => less than a fraction 10^{-6} of entangled states not detected as entangled.

Non-symmetric states

$$\mathcal{M} = \{x_1, y_1, z_1, x_2, x_1x_2, y_1x_2, z_1x_2, y_2, x_1y_2, y_1y_2, z_1y_2, z_2, x_1z_2, y_1z_2, z_1z_2\}$$

- Number m_k of inequivalent measurement sets and size of moment matrices grow quickly

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
m_k	3	10	30	69	132	205	254	254	205	132	69	30	10	3	1

- Calculated probabilities of sets with k<6
  - No state detected with only one measurement
  - Biggest fraction of states detected by

$$\begin{aligned}
\{x_1x_2, y_1y_2\} &\quad\rightarrow\quad p^{(2)} \simeq 1\% \\
\{x_1x_2, y_1y_2, z_1z_2\} &\quad\rightarrow\quad p^{(3)} \simeq 10\% \\
\{x_1x_2, x_1y_2, y_1x_2, z_1z_2\} &\quad\rightarrow\quad p^{(4)} \simeq 12\% \\
\{x_1x_2, x_1y_2, y_1x_2, y_1y_2, z_1z_2\} &\quad\rightarrow\quad p^{(5)} \simeq 23\%
\end{aligned}$$

IV. Separability of quantum channels

Quantum Channels

$$\Phi : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$$

$$\rho' = \Phi(\rho)$$

Choi matrix

$$C_\Phi = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

- Linear operator on $H \otimes H'$, $H'(= H)$ = ancilla
- $C_\Phi \geq 0 \Leftrightarrow \Phi$ completely positive
- C_Φ / N = state on $H \otimes H'$

Separability of channels

$$H = H_A \otimes H_B \Rightarrow \Phi \text{ linear operator on } \mathcal{H} \equiv H_A \otimes H_B \otimes H_{A'} \otimes H_{B'}$$

For any positive operator, M is called separable iff $M = \sum_k P_k \otimes Q_k, \quad P_k, Q_k \geq 0$

- Separable channel $\quad \Phi_{\text{sep}} \Leftrightarrow E_l = A_l \otimes B_l$
  - Factorizing Kraus operators
  - Maps separable states to separable states
  - Choi matrix separable across (A-A') – (B-B') cut

N. Johnson, PhD thesis, U Guelph (2012)

J.I.Cirac et al, PRL 2001

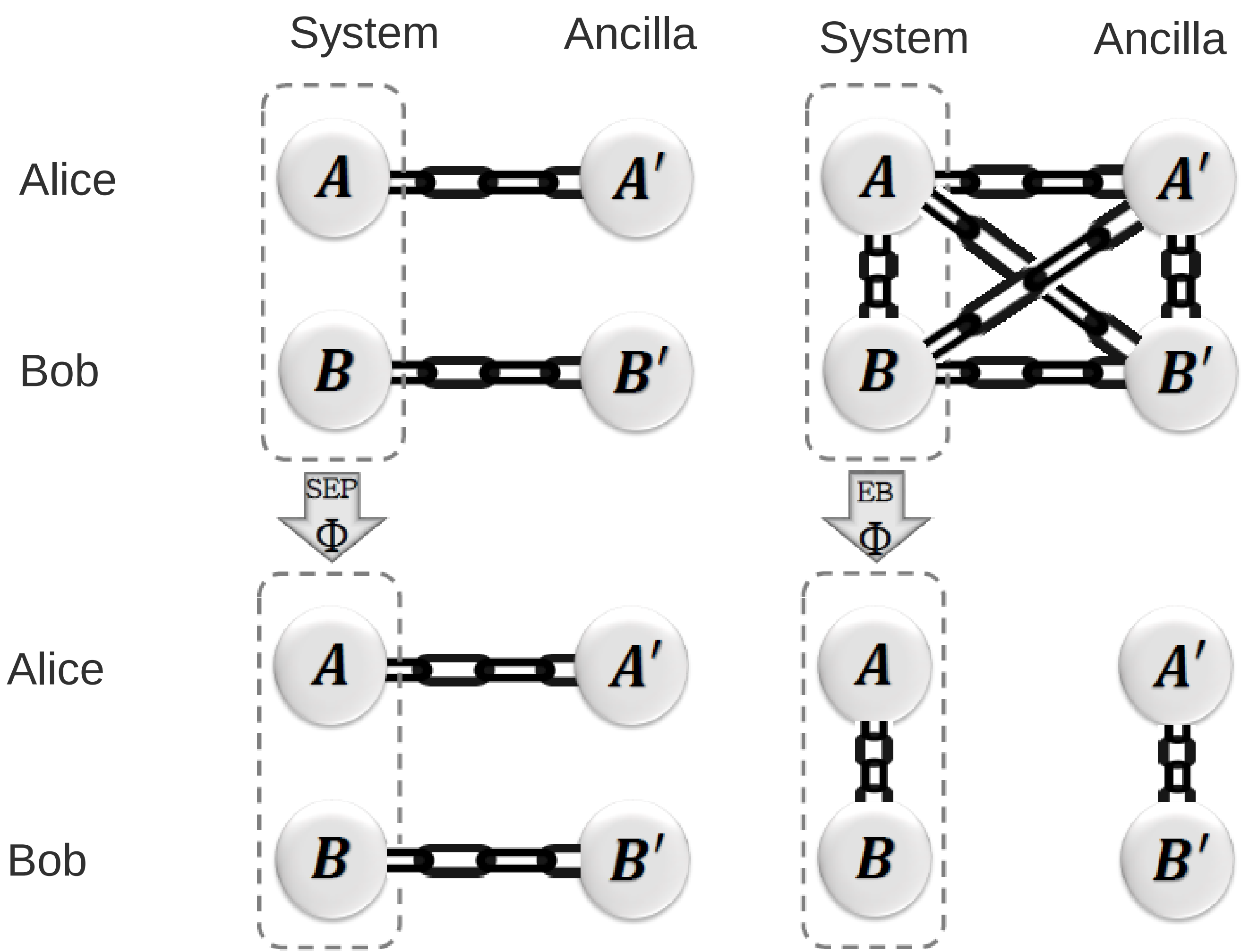
- Entanglement breaking channel

M. Horodecki, P. Shor, M.B. Ruskai, Rev.Math.Phys. 2003

$$\Phi_{\text{EB}} \Leftrightarrow (\Phi \otimes \mathbf{1})\rho \text{ separable across } H - H' \text{ cut } \forall \ \rho \in \mathcal{L}(\mathcal{H})$$

  - Choi matrix separable across (A-B) – (A'-B') cut
  - Defined even for a single system A

Separability of channels



Semialgebraic set

$$C_\Phi = \sum_k P_k \otimes Q_k$$

$$P_k = \sum_\lambda c_\lambda^{(k)} S_\lambda^{AB}$$

$$Q_k = \sum_{\lambda'} d_\lambda^{(k)} S_\lambda^{A'B'}$$

Basis of hermitian, orthogonal operators

For EB
Similarly for SEP

Positivity
=> inequalities on the coefficients of the characteristic polynomials by Decartes' sign rule
=> polynomial inequalities in c_λ and d_λ.

EB for channels

Case of EB channels

$$C_\Phi = \sum_{\lambda,\lambda'} X_{\lambda\lambda'} S_\lambda^{AB} \otimes S_{\lambda'}^{A'B'}$$

$$\Rightarrow X_{\lambda\lambda'} = \sum_k c_\lambda^{(k)} d_{\lambda'}^{(k)} = \int d\mu(x) x_\lambda x_{\lambda'} \Rightarrow \text{a tms } (y_\alpha)_{\alpha\leq 2}$$

Theorem:

The channel Φ is EB iff $\exists$ a flat extension $(y_\beta)_{\beta\leq 2(t+d_0)}$ of $(y_\beta)_{\beta\leq 2t}$ $(t \geq 1)$, with $M_t(y) \geq 0$ and $M_t(g_j \star y) \geq 0$ for $j = 1,\dots,m$, where the g_j are polynomials of variables c_λ and $d_{\lambda'}$ defined by the conditions $\sum_\lambda c_\lambda S_\lambda^{AB} \geq 0$, $\sum_{\lambda'} d_{\lambda'} S_{\lambda'}^{A'B'} \geq 0$, and $d_0 = \max_{1\leq j\leq m}\{1, \lceil \deg(g_j)/2 \rceil\}$.

Similarly for SEP.

Scaling

- 2 qubit channel
- 15 parameters x_μ for each subsystem ((A-B) and (A'-B') for EB)
- $\binom{30+2t}{2t}$ free variables in moment matrix of order-t extension
- Semialgebraic set defined by polynomials of degree 4, hence $d_0=2$
- Smallest moment matrix containing all moments given by quantum channel is M_1, smallest extension is M_3 with size $\binom{33}{3} = 5456$, containing $\binom{36}{6} > 10^6$ free variables

=> restrict to less general channels

EB in a qutrit channel

$$\rho = \frac{1}{3}(\mathbb{1} + \sum_{i=1}^{8} \zeta_i \lambda_i)$$ ← Gell-Mann matrices

- Family of damping channels
- Affine trafo of (generalized) Bloch vector ζ $\quad \Phi_D : \zeta \to \zeta' = \Lambda\zeta$

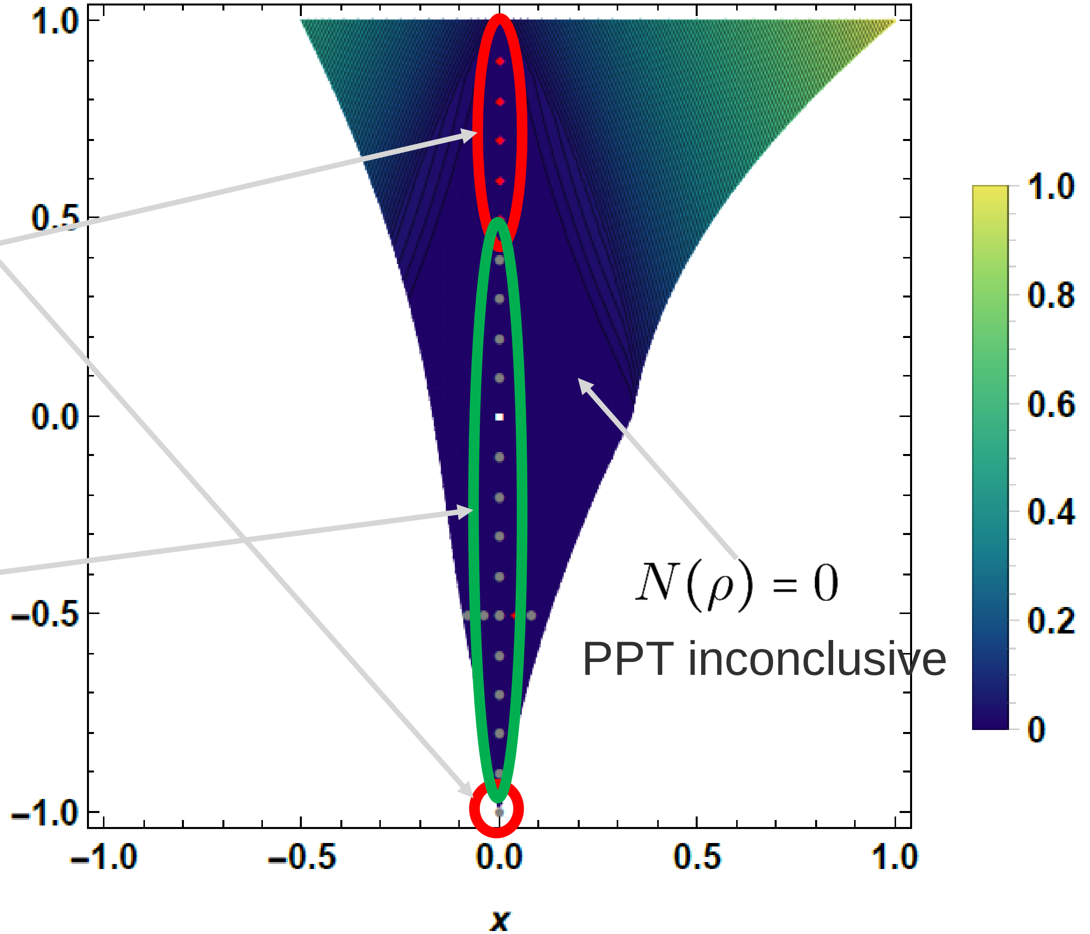
$$\Lambda = \mathrm{diag}(0, 0, x, 0, 0, 0, 0, y^2)$$

- If negativity $N(\rho) = \frac{1}{2}(\|\rho^{T_H}\|_1 - 1) > 0$, channel is not separable
- PPT entangled Choi states may exist for $N(\rho) = 0$

Qutrit channel

tms algo inconclusive at reachable extension order

tms algo gives certificate of SEP



Negativity

N. Milazzo, DB, O. Giraud, PRA 2020 (to appear); arXiv:2006.15003 [quant-ph]

Summary

- tms algorithm completely solves serapability problem for states and channels
- Generalizes and extends previous results by Doherty et al. [Doherty, Parillo, Spedallieri PRL 2002]
  - unified mathematicle framework
  - accomodates missing data, different dimensions, symmetries,…
- Tool for studying optimal sets and sequences of measurements
- Numerical solution by SDP, but limited to relatively small system sizes

Most importantly…

PhD position available in q-metrologie !

- Work with
  - Nadia Milazzo (PhD work)
  - Fabian Bohnet-Waldraff (PhD work)
  - Olivier Giraud, LPTMS & CNRS Paris-Saclay

  - Thierry Bastin
  - John Martin
  - Dorian Baguette

  } Université de Liège

- Support: Deutsch-Französische Hochschule (UFA), grant CT-45-14-II/2015

Université franco-allemande Deutsch-Französische Hochschule

O. Giraud, DB, D. Baguette, T. Bastin, and J. Martin, PRL 2015
F. Bohnet-Waldraff, DB, Giraud, PRA 2016, 2017
N. Milazzo, DB, O. Giraud, PRA 2019
N. Milazzo, DB, O. Giraud, PRA 2020 (to appear)

Backup

Code

```
mpol x 3
xMom=[1 x(1) x(2) x(3) x(1)^2 x(1)*x(2)
    x(1)*x(3) x(2)^2 x(2)*x(3) x(3)^2];
y=[X(1,1) X(1,2) X(1,3) X(1,4) X(2,2) X
    (2,3) X(2,4) X(3,3) X(3,4) X(4,4)];
con=[mom(xMom)==y];
K = [x(1)^2+x(2)^2+x(3)^2-1==0];
G = randn(length(xMom));
R = xMom*(G'*G)*xMom'; k=2;
P = msdp(min(mom(R)),K,con,k);
pars.eps=0; mset(pars);
[status] =msol(P);
```

Weinberg matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$

- Overcomplete set of matrices: Expand (square of) Lorentz boost operator in powers of **q** and identify terms $q_{\mu 1}q_{\mu 2}\ldots q_{\mu N}$

S. Weinberg, PR 1964

$$\Pi^{(j)}(q) \equiv (q_0^2 - |\mathbf{q}|^2)^j \, e^{-2\eta_q \, \hat{\mathbf{q}}\cdot\mathbf{J}}$$

$$\Pi^{(j)}(q) = (-1)^{2j} q_{\mu_1} q_{\mu_2} \ldots q_{\mu_{2j}} S_{\mu_1\mu_2\ldots\mu_{2j}}$$

w/ Einstein summation Convention, μ_i=0,1,2,3
j=N/2

- E.g. spin-1/2 (N=1):

$$\Pi^{(1/2)}(q) = -q_0 - 2\mathbf{q}\cdot\mathbf{J}$$

So $S_0 = \sigma_0$ and $S_a = 2J_a = \sigma_a$

$$\rho = \frac{1}{2} x_{\mu_1} S_{\mu_1}$$

Bloch sphere picture!

- Spin-1 (N=2):

$$\Pi^{(1)}(q) = (q_0^2 - \mathbf{q}^2) + 2\mathbf{q}\cdot\mathbf{J}\,(\mathbf{q}\cdot\mathbf{J} + q_0) = q_{\mu_1} q_{\mu_2} S_{\mu_1\mu_2}.$$

$$S_{00} = J_0, \; S_{a0} = J_a \text{ and } S_{ab} = J_a J_b + J_b J_a - \delta_{ab} J_0$$

$$\rho = \frac{1}{4} x_{\mu_1\mu_2} S_{\mu_1\mu_2}$$

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Properties of Weinberg matrices

- 4^N Hermitian matrices (overcomplete set!)
- Traceless in the relativistic sense $g_{\mu_1\mu_2} S_{\mu_1\mu_2\ldots\mu_{2j}} = 0, \quad g \equiv \mathrm{diag}(-,+,+,+)$

Theorem: The Weinberg matrices $S_{\mu 1\mu 2\ldots\mu N}$ are given by the projection of tensor products of Pauli matrices into the subspace $\mathcal{H}_S$ of states that are invariant under permutation of particles.

$$\langle D_N^{(k)} | S_{\mu_1\mu_2\ldots\mu_N} | D_N^{(\ell)} \rangle = \langle D_N^{(k)} | \boldsymbol{\sigma}_{\mu_1} \otimes \boldsymbol{\sigma}_{\mu_2} \otimes \ldots \otimes \boldsymbol{\sigma}_{\mu_N} | D_N^{(\ell)} \rangle$$

$$|D_N^{(k)}\rangle = \mathcal{N} \sum_\pi |\underbrace{0\ldots0}_{N-k}\underbrace{1\ldots1}_{k}\rangle, \quad k = 0,\ldots,N$$

Symmetric Dicke states of *N* two level systems with *k* excitations

Proof: use SU(2) disentangling theorem and SU(2) coherent state representation

Corrollary 1: The Weinberg matrices form a 2^N – **tight frame**.

Corrollary 2:

$$\rho = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N} S_{\mu_1\mu_2\ldots\mu_N}$$

$$x_{\mu_1\mu_2\ldots\mu_N} = \mathrm{tr}(\rho\, S_{\mu_1\mu_2\ldots\mu_N})$$

Bloch **tensor** picture for a spin-j, j=N/2

O. Giraud, DB, D. Baguette, T. Bastin, and J. Martin, PRL 2015

Tight frames

A family of vectors $|\phi_i\rangle$, $i \in \{1, \ldots, M\}$, is called a frame for a Hilbert space $\mathcal{H}$ with bounds $A, B \in]0, \infty[$, if

$$A||\psi||^2 \leqslant \sum_{i=1}^{M} |\langle\psi|\phi_i\rangle|^2 \leqslant B||\psi||^2, \quad \forall\ |\psi\rangle \in \mathcal{H}.$$

If $A = B$, then the frame is called an A-tight frame.

Properties of Bloch tensor

- Rotation under SU(2) transformation:

$$x_{\mu_1 \dots \mu_N} \to R_{\mu_1 \nu_1} \dots R_{\mu_N \nu_N} x_{\nu_1 \dots \nu_N}$$

generalizes rotation of Bloch vector: $x_a \to R_{ab} x_b$

- Coordinates of SU(2) coherent state pointing in direction **n**:

$$x_{\mu_1 \mu_2 \dots \mu_N} = n_{\mu_1} n_{\mu_2} \dots n_{\mu_N}$$

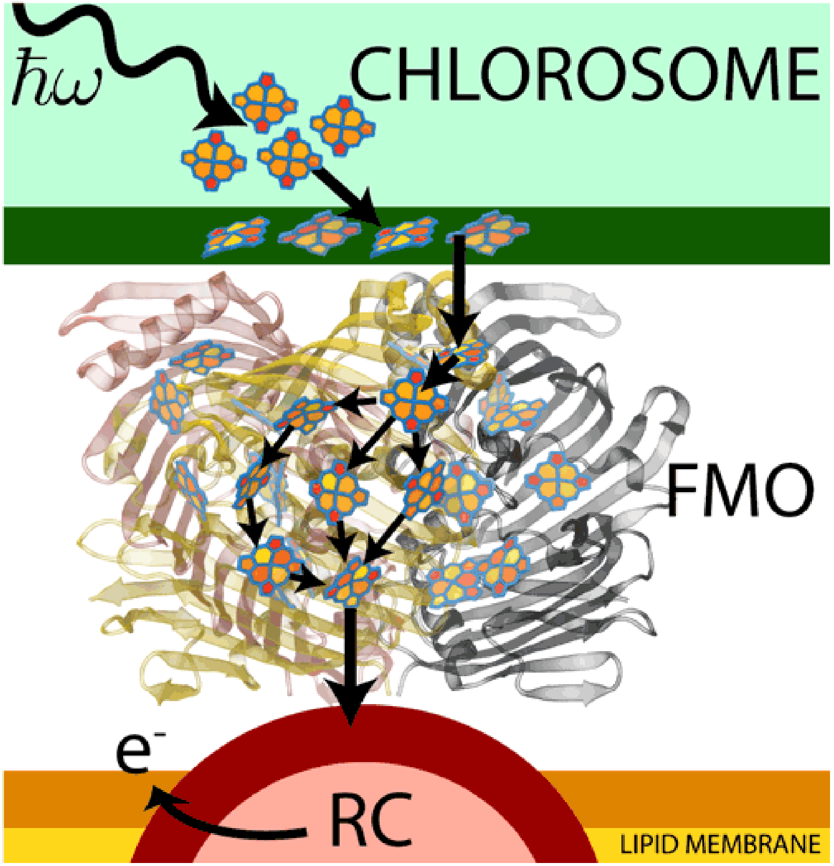
- Spin-k reduced density matrix for symmetric state of a multi-qubit system:

$$x_{\mu_1 \dots \mu_{2k}} = x_{\mu_1 \dots \mu_{2k} 0 \dots 0}$$

- Scalar product

$$\mathrm{tr}(\rho \rho') = \frac{1}{2^N} x_{\mu_1 \mu_2 \dots \mu_N} x'_{\mu_1 \mu_2 \dots \mu_N},$$

Spin =! Pseudo spin



- Quantumness of energy transfer in photosynthesis in *Chlorobaculum tepidem*
- 7-state system
- Which two states to choose as |j,-j>, |j,j> (i.e. “most classical states”)?

=> Pointer states of energy-current!

Tensor rep and positive partial transpose

- T matrix: $$T_{\boldsymbol{\mu},\boldsymbol{\nu}} = X_{\mu_1\ldots\mu_j\nu_1\ldots\nu_j}$$

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Tensor rep and positive partial transpose

- T matrix: $$T_{\boldsymbol{\mu},\boldsymbol{\nu}} = X_{\mu_1\ldots\mu_j\nu_1\ldots\nu_j}$$
- Theorem: ρ^{T_A} and T are similiar, i.e.

$$\exists R \text{ unitary and } \lambda > 0 \mid R^\dagger \rho^{T_A} R = \lambda T$$

Bohnet-Waldraff, DB, O.Giraud, PRA ‘16

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Bohnet-Waldraff, DB, O.Giraud, PRA '16

- Constructive proof:
  - 1. Coherent state (P-)rep $$\rho = \int d\alpha P(\alpha)|\alpha\rangle\langle\alpha| d\alpha$$
  - 2. Explicit *R*: $$R_{\boldsymbol{i},\boldsymbol{\mu}} = \frac{1}{2^{j/2}} \prod_{k=1}^{j} \sigma^{\mu_k}_{i_k, i_{k+j}}$$ $\boldsymbol{i} = (i_1 i_2 \ldots i_N)$ and $\boldsymbol{\mu} = (\mu_1\mu_2\ldots\mu_j)$, $0 \le \mu_k \le 3$ and $0 \le i_k \le 1$

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Bohnet-Waldraff, DB, O.Giraud, PRA '16

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- Corollary $$\rho^{T_A} \ge 0 \iff T \ge 0$$

Tensor rep and positive partial transpose

- Simplification: remove redundant lines and columns from T; positivity unchanged; sufficient condition for separability directly from (N+1)x(N+1) real Hermitian matrix T
- Unifies several previous criteria
  - N=2 $(X_{\mu,\nu})_{0\leq\mu,\nu\leq 3} \geq 0$ O. Giraud, P. Braun, DB PRA '08
  - Hierarchy of PPT criteria from correlation matrices of reduced ρ:

$$C^{(r)}_{\boldsymbol{\mu}_r,\boldsymbol{\nu}_r} = X_{\boldsymbol{\mu}_r\boldsymbol{\nu}_r\mathbf{0}_{N-2r}} - X_{\boldsymbol{\mu}_r\mathbf{0}_{N-r}}X_{\boldsymbol{\nu}_r\mathbf{0}_{N-r}}$$

$$\rho_r^{T_A} \geq 0 \iff C^{(r)} \geq 0$$

Devi, Prabhu, Rajagopal PRL'07
Tóth & Gühne, PRL '09

$$C^{(r)} \to \text{Schur complement of } T^{(r)}$$

- $R_{i\mu}$ generalizes "magic basis" known for N=2 Hill & Wootters, PRL '97

Tensor rep and positive partial transpose

- PPT criterion

$$\rho \text{ separable} \implies \rho^{T^A} = \sum_i p_i \underbrace{\rho_A^{(i)^T}}_{\geq 0} \otimes \rho_B^{(i)} \geq 0$$

$$\Rightarrow \rho^{T_A} \ngeq 0 \implies \rho \text{ entangled}$$

Peres PRL'96; M,P,R Horodecki '96

$$\impliedby \; 2 \times 2, \;\; 2 \times 3$$

$$\not\Longleftarrow \;\; \text{higher dim: ''bound entanglement''}$$

P. Horodecki '97; Amselem,Bourennane '09