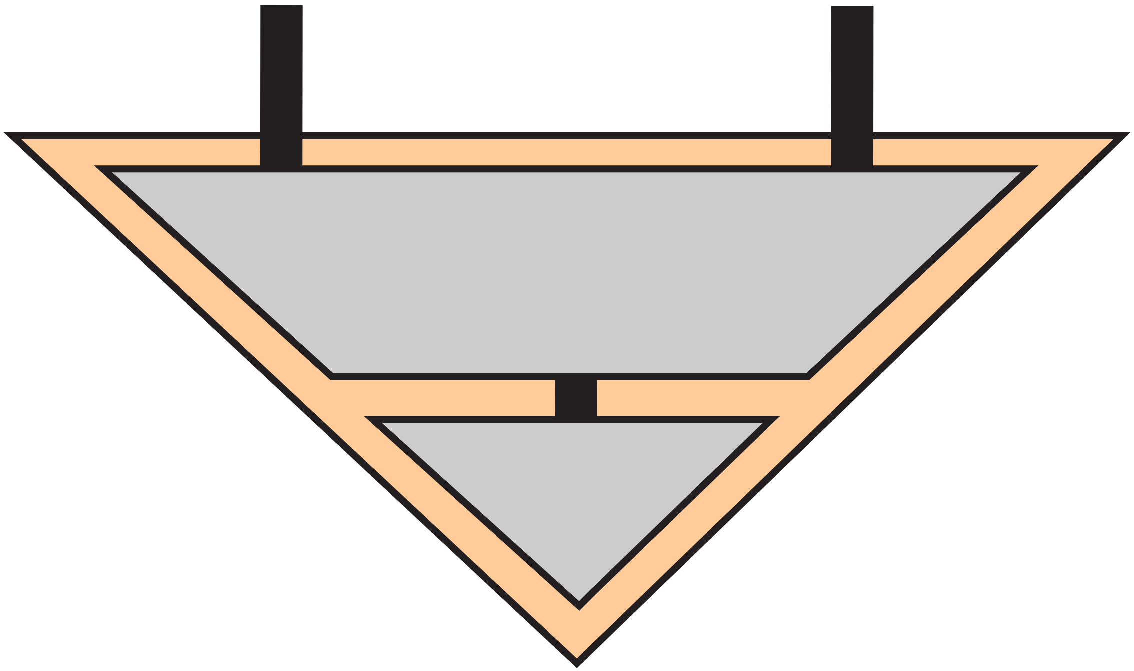
Bob Coecke
University of Oxford



Kindergarten Quantum Mechanics

— beyond the Hilbert space formalism —

Quantum informatics context of this work

“What is the true origin of quantum algorithmic speed-up?”

“How do quantum and classical information interact?”

“What are the limits of quantum computation?”

What is a convincing model thereof?”

“What are the foundational structures of QIC?”

Foundational Structures for QIC

— FET Open (2006) EC STREP-network —

Bristol	Richard Jozsa
Braunschweig	Reinhard Werner
Grenoble	Philippe Jorrand
Innsbruck	Hans Briegel
McGill	Prakash Panangaden
Paris 7	Vincent Danos
Oxford	Samson Abramsky
York	Sam Braunstein
Coordination	Bob Coecke

Ended 2nd out of 500 submissions for an FP6 open call!

Our approach: rebuild QM from scratch!

Kinds/types of systems:

$$A\,,\ B\,,\ C\,,\ ...$$

- e.g. electron, atom, n qubits, classical data, ...

Kinds/types of systems:

$$A\ ,\ B\ ,\ C\ ,\ ...$$

- e.g. electron, atom, n qubits, classical data, ...

Operations/experiments on systems:

$$A \xrightarrow{f} A\ ,\quad A \xrightarrow{g} B\ ,\quad B \xrightarrow{h} C\ ,\ ...$$

- e.g. preparation, acting force field, measurement, ...

Kinds/types of systems:

$$A \, , \, B \, , \, C \, , \, ...$$

- e.g. electron, atom, n qubits, classical data, ...

Operations/experiments on systems:

$$A \xrightarrow{f} A \, , \;\; A \xrightarrow{g} B \, , \;\; B \xrightarrow{h} C \, , \, ...$$

- e.g. preparation, acting force field, measurement, ...

Sequential composition of operations:

$$A \xrightarrow{h \circ g} C \quad := \quad A \xrightarrow{g} B \xrightarrow{h} C \qquad\qquad A \xrightarrow{1_A} A$$

Kinds/types of systems:

$$A\,,\ B\,,\ C\,,\ \ldots$$

- e.g. electron, atom, n qubits, classical data, ...

Operations/experiments on systems:

$$A \xrightarrow{f} A\,,\quad A \xrightarrow{g} B\,,\quad B \xrightarrow{h} C\,,\ \ldots$$

- e.g. preparation, acting force field, measurement, ...

Sequential composition of operations:

$$A \xrightarrow{h\circ g} C \quad := \quad A \xrightarrow{g} B \xrightarrow{h} C \qquad\qquad A \xrightarrow{1_A} A$$

Multiplicity of systems/operations:

$$A \otimes B \qquad\qquad A \otimes C \xrightarrow{f\otimes g} B \otimes D$$

+ “obvious” rules governing $\circ$-$\otimes$ interaction

+ “obvious” rules governing $\circ$-$\otimes$ interaction

$$\boxed{= \text{tensor category}}$$

+ “obvious” rules governing $\circ$-$\otimes$ interaction

= tensor category

There are graphical calculi comprising these!

+ “obvious” rules governing $\circ$-$\otimes$ interaction

$\boxed{= \text{tensor category}}$

There are graphical calculi comprising these!

graphical language for $\otimes$-categories:

$\otimes \sim \textit{horizontal} \quad \circ \sim \textit{vertical}$

+ “obvious” rules governing $\circ$-$\otimes$ interaction

$= $ tensor category

There are graphical calculi comprising these!

graphical language for $\otimes$-categories:

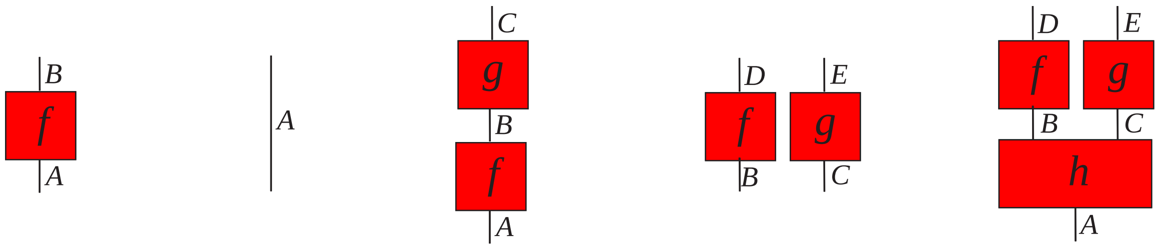
$\otimes \sim$ *horizontal* $\quad \circ \sim$ *vertical*

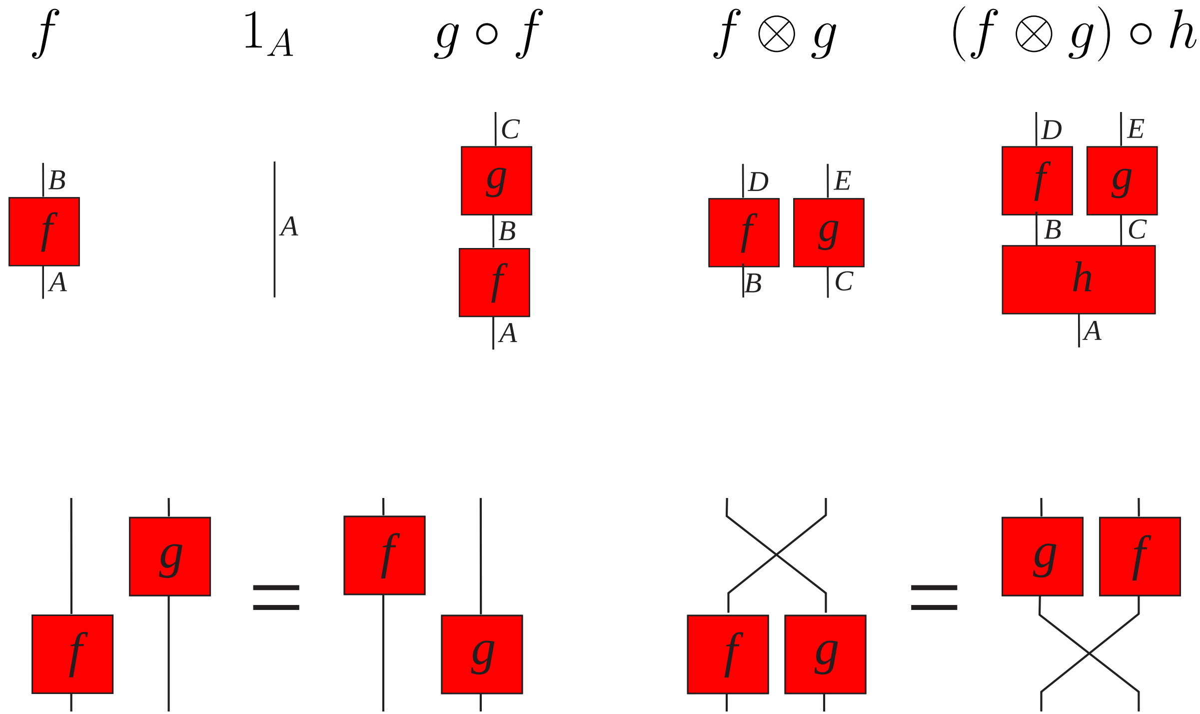
provable from categorical axioms

$\Longleftrightarrow$

derivable in graphical language

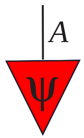
f $\qquad 1_A$ $\qquad g \circ f$ $\qquad f \otimes g$ $\qquad (f \otimes g) \circ h$

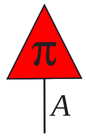


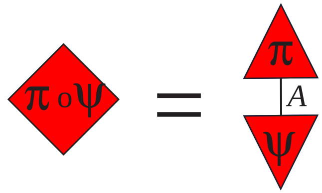


f
1_A
$g \circ f$
$f \otimes g$
$(f \otimes g) \circ h$
B
f
A
A
C
g
B
f
A
D
E
f
g
B
C
D
E
f
g
B
C
h
A
g
f
=
f
g
f
g
=
g
f

$$\psi : \mathrm{I} \longrightarrow A \qquad \pi : A \longrightarrow \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$$



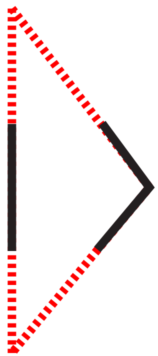




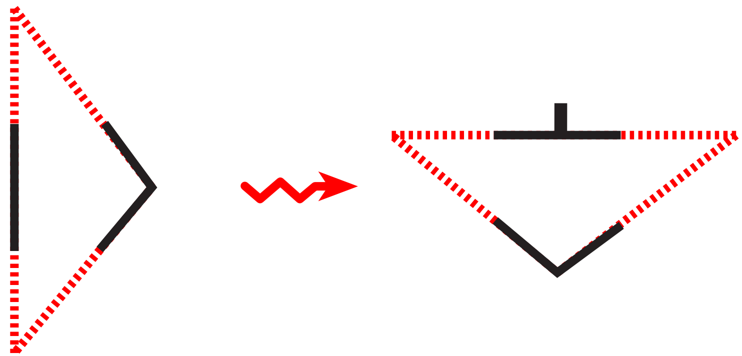
$\psi : \mathrm{I} \to A$ $\quad$ $\pi : A \to \mathrm{I}$ $\quad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$

$$| \; \rangle$$

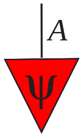
$$\psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$$

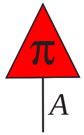


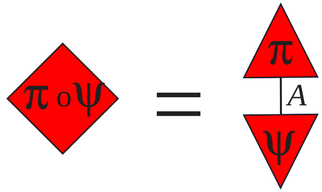
$\psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$



$\psi : \mathrm{I} \to A$ $\qquad$ $\pi : A \to \mathrm{I}$ $\qquad$ $\pi \circ \psi : \mathrm{I} \to \mathrm{I}$

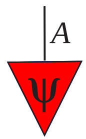


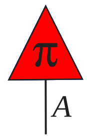


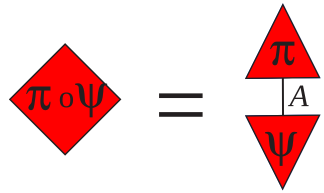


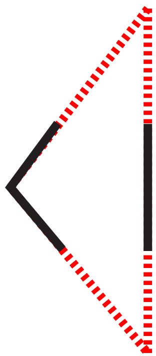
$\langle \; |$

$\psi : \mathrm{I} \longrightarrow A$ $\pi : A \longrightarrow \mathrm{I}$ $\pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$

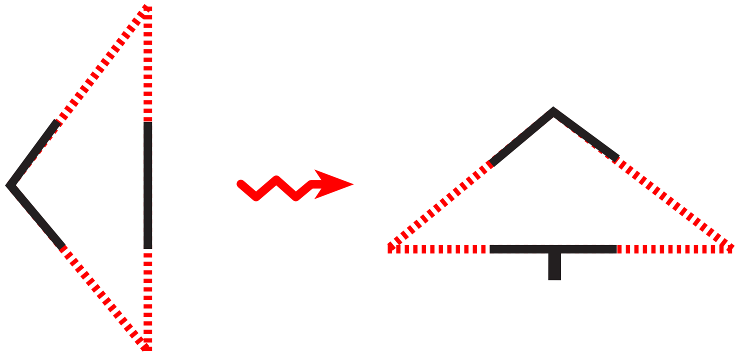




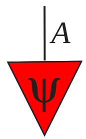




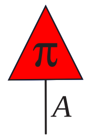
$$\psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$$



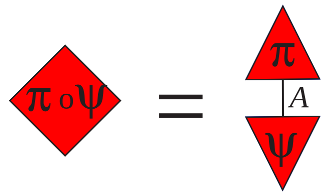
$\psi : \mathrm{I} \to A$



$\pi : A \to \mathrm{I}$

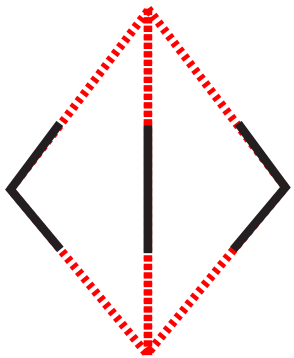


$\pi \circ \psi : \mathrm{I} \to \mathrm{I}$

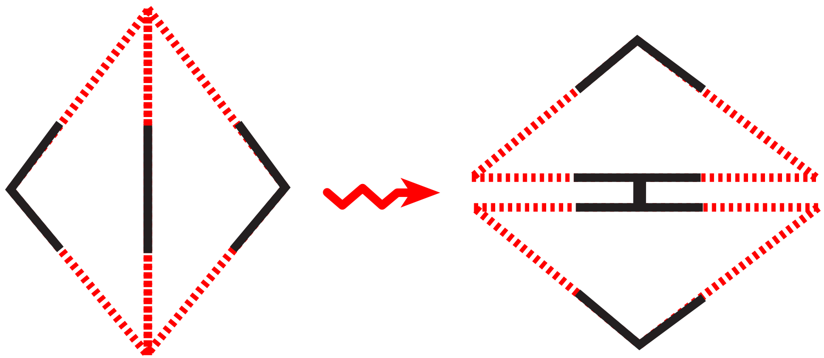


$$\langle \quad | \quad \rangle$$

$\psi : \mathrm{I} \longrightarrow A$ $\pi : A \longrightarrow \mathrm{I}$ $\pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$



$\psi : \mathrm{I} \longrightarrow A \qquad \pi : A \longrightarrow \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$



$$f : A \to B \quad \longleftrightarrow \quad f^\dagger : B \to A$$

QUANTUM STRUCTURE

Abramsky-Coecke (2004) IEEE-LiCS

Kelly-Laplaza (1980) ***Coherence for compact closed categories*****.**

Selinger (2007) †-***Compact categories and CPMs*****.**

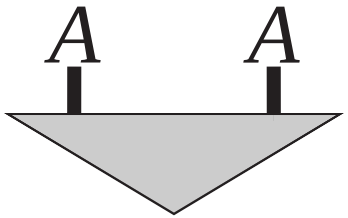
Empirical fact: entangled states exist in nature

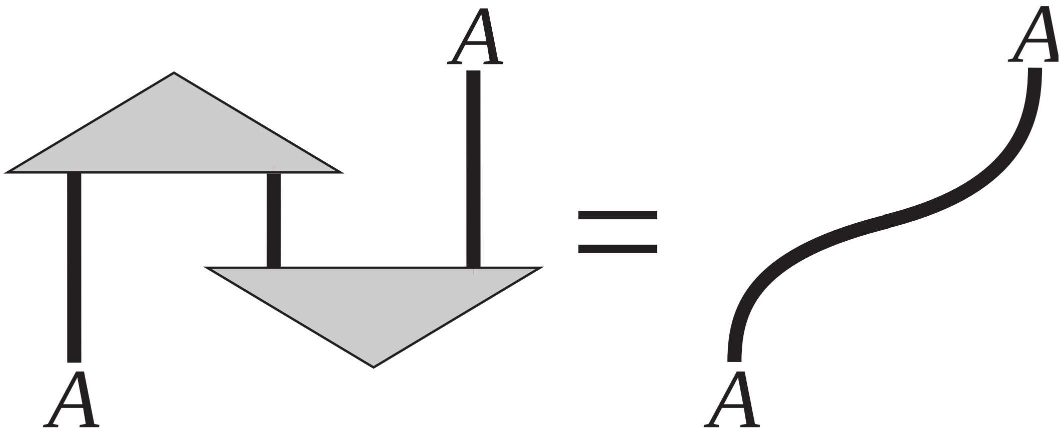
Empirical fact: entangled states exist in nature

Quantum structure :=

Bell-states exist + their behaviour

System with quantum structure





System with quantum structure

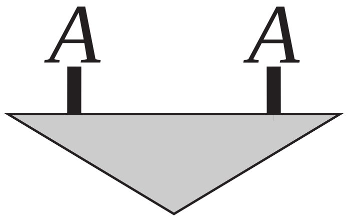
A pair

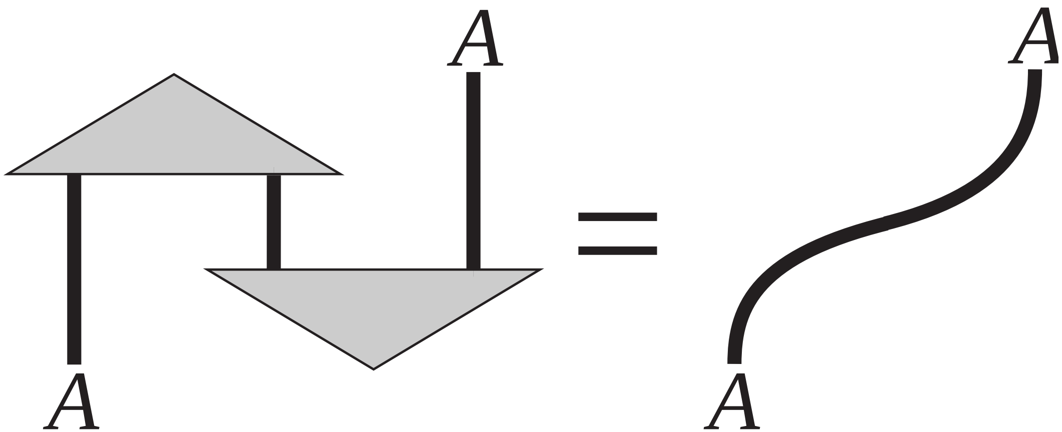
$$(A\,, \eta : \mathrm{I} \to A \otimes A)$$

such that:

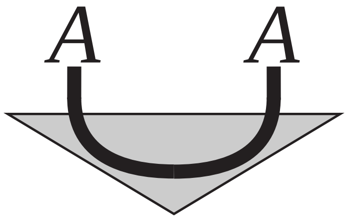
$$\begin{array}{ccccc}
A & \xleftarrow{\ \simeq\ } & \mathrm{I} \otimes A & \xleftarrow{\ \eta^\dagger \otimes 1_A\ } & (A \otimes A) \otimes A \\
\uparrow{\scriptstyle 1_A} & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\ \simeq\ ]{} & A \otimes \mathrm{I} & \xrightarrow[\ 1_A \otimes \eta\ ]{} & A \otimes (A \otimes A)
\end{array}$$

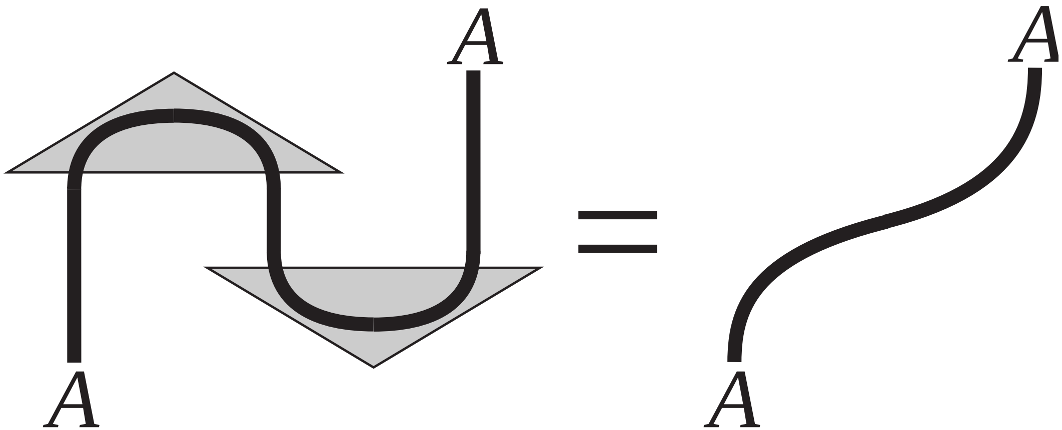
System with quantum structure



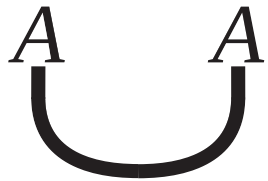


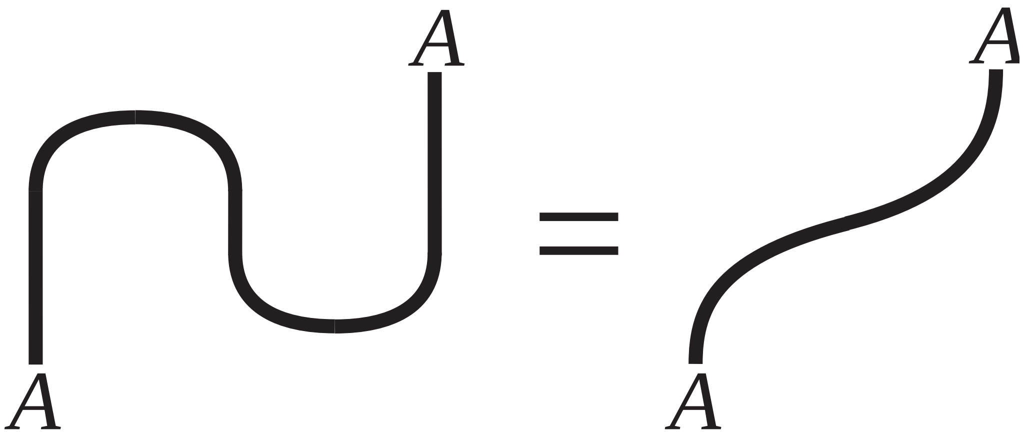
System with quantum structure

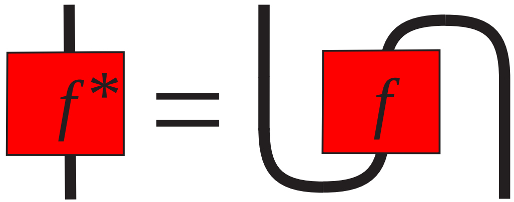


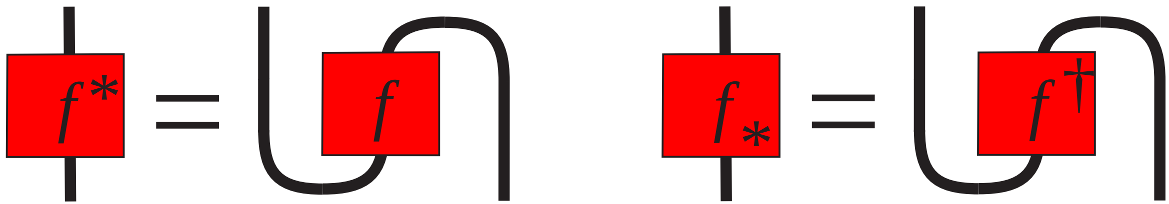


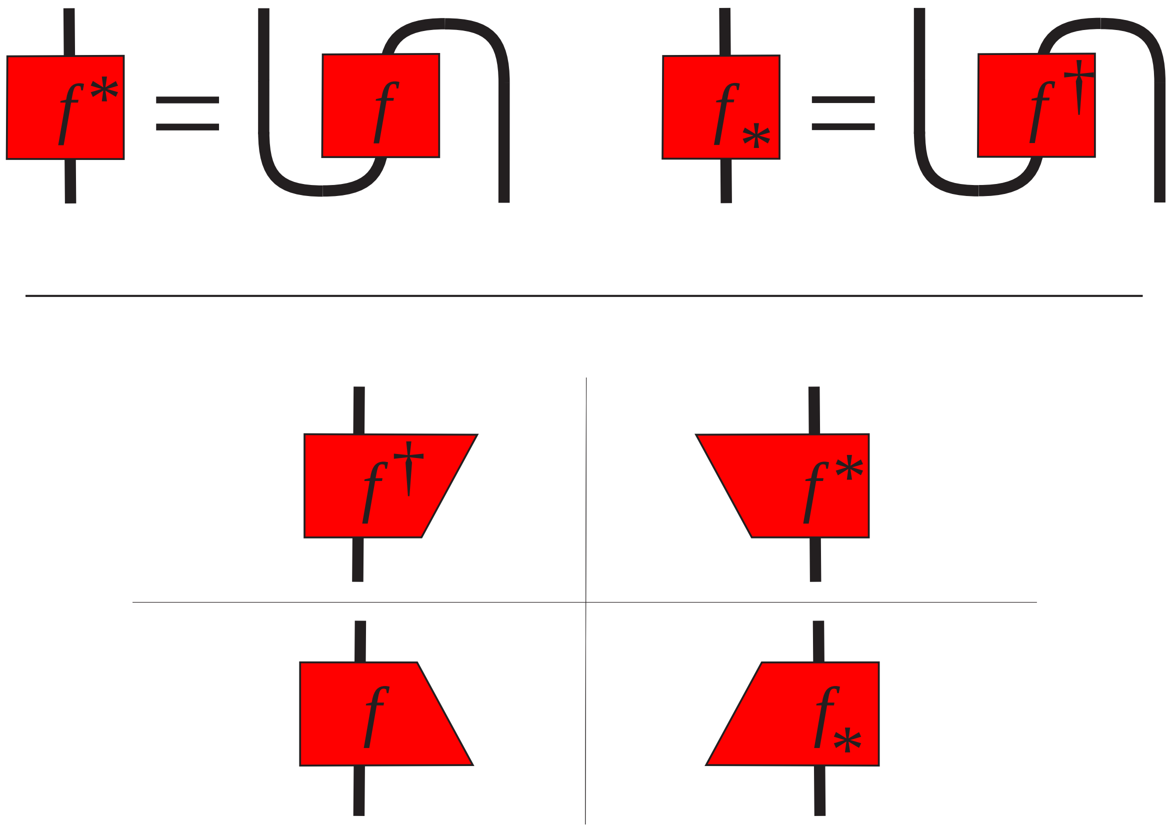
System with quantum structure



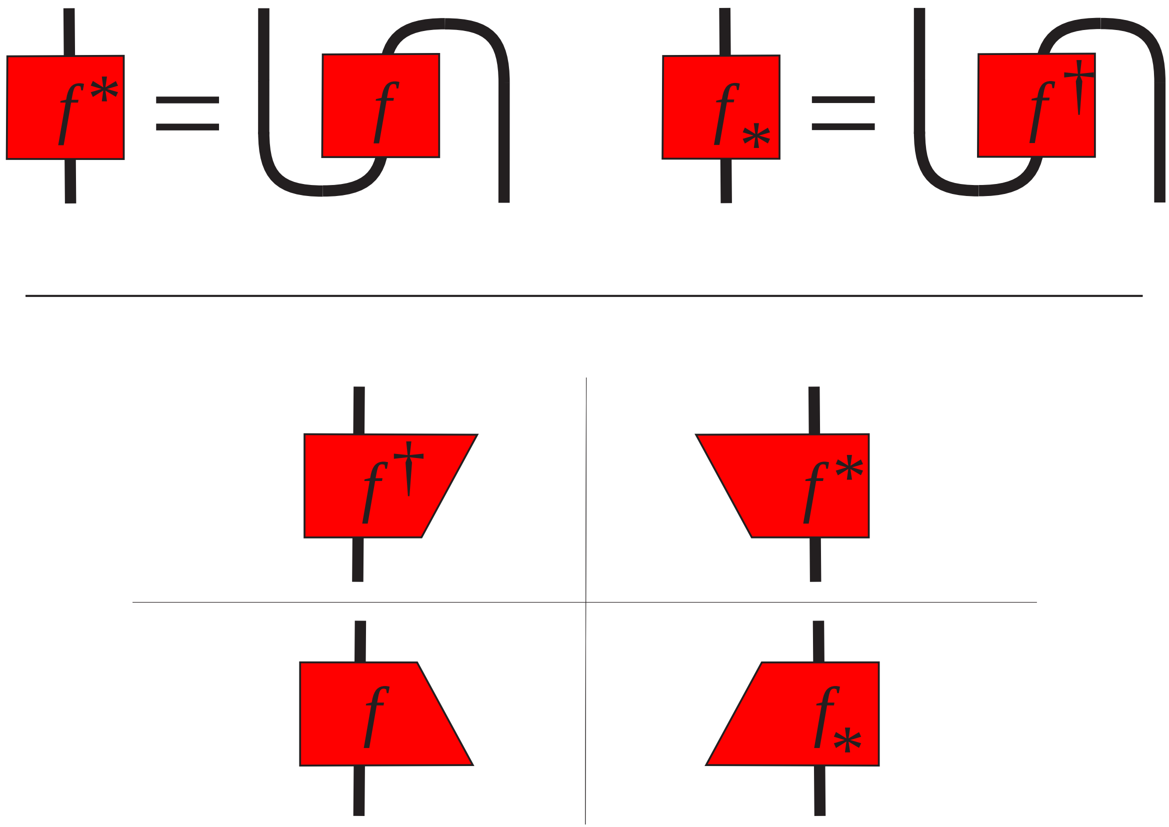



f*
=
f

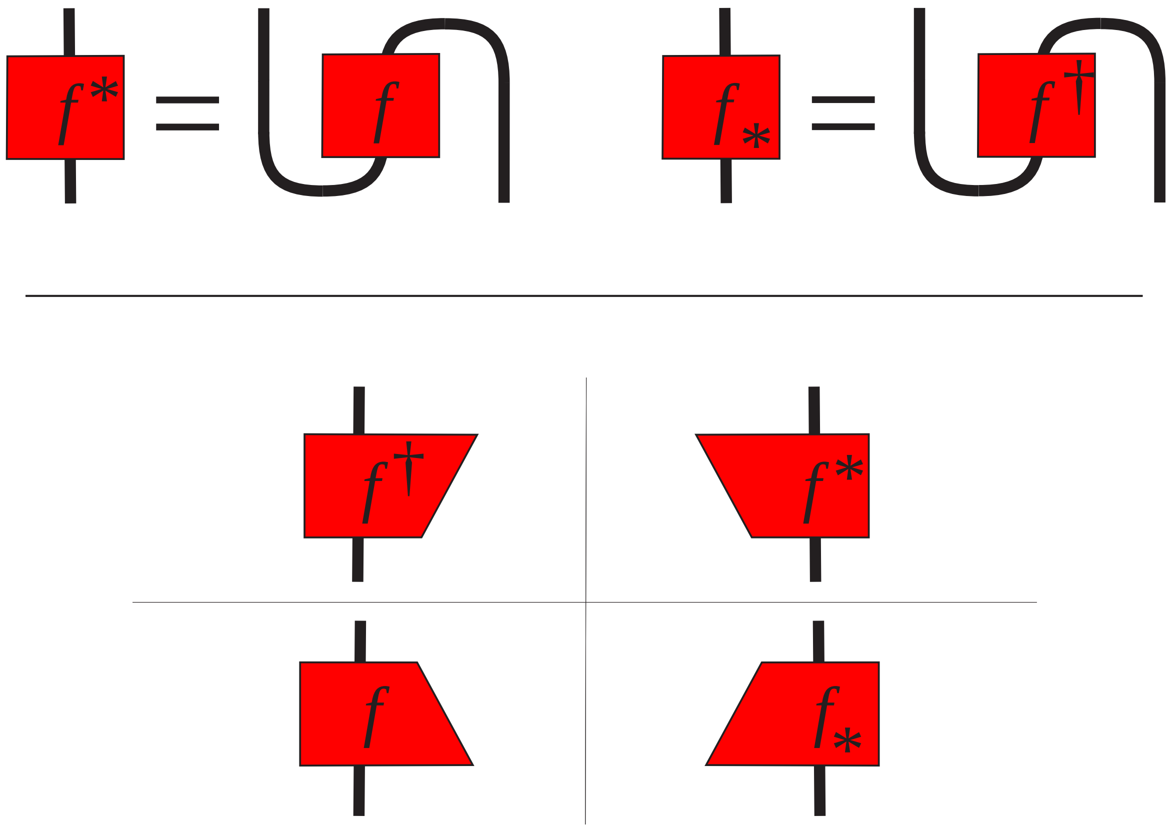

f*
=
f
f*
=
f†



Graphical representation captures their relations

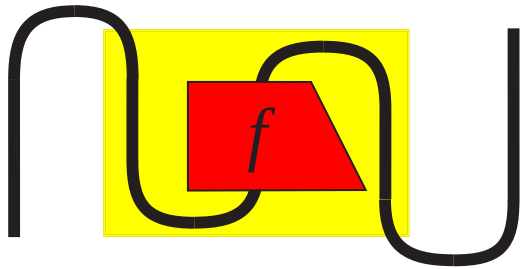


$$(f^*)_* = f^\dagger$$

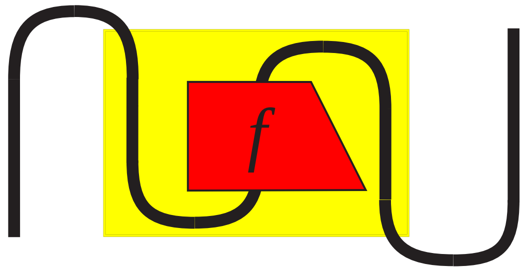


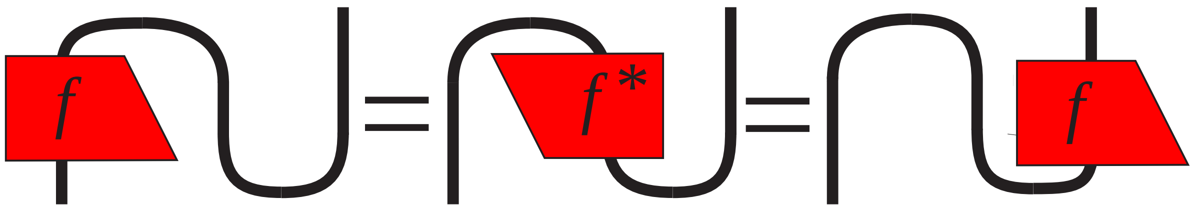
In Hilb: $f^* \sim$ transposed & $f_* \sim$ conjugated

“Sliding” boxes

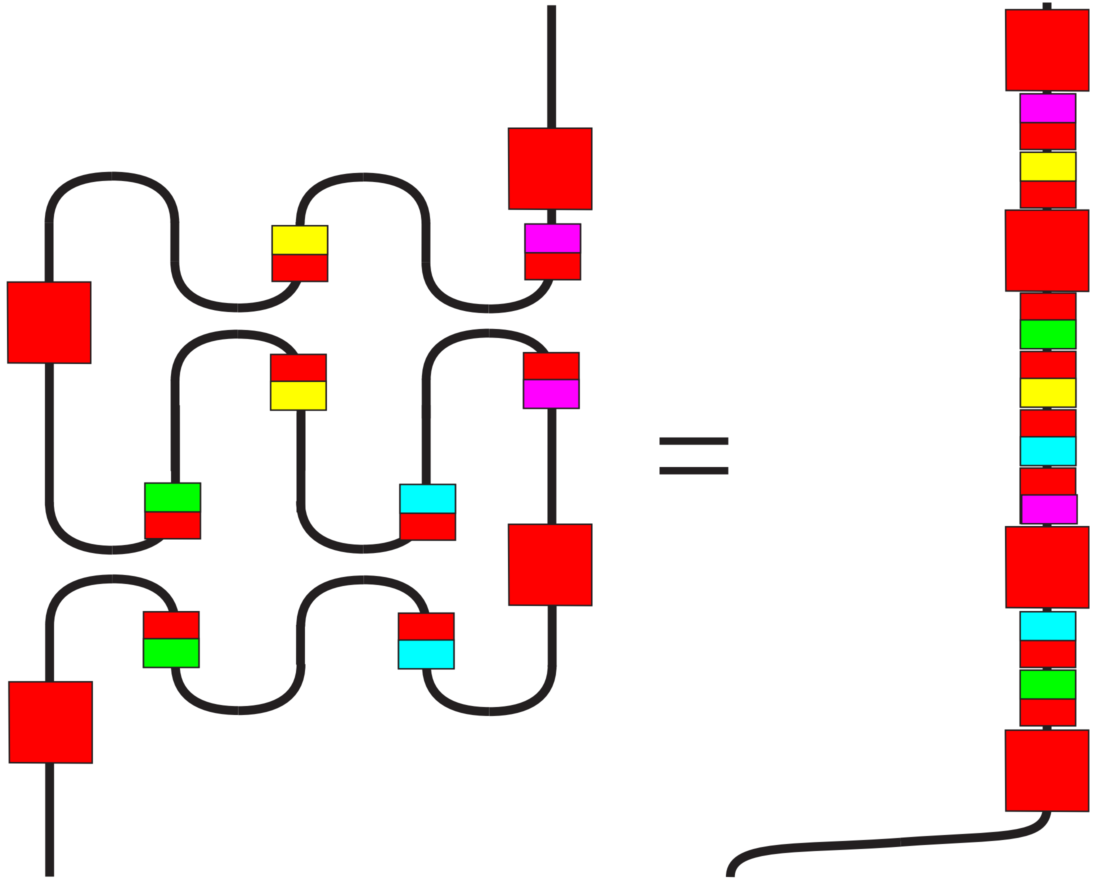


“Sliding” boxes

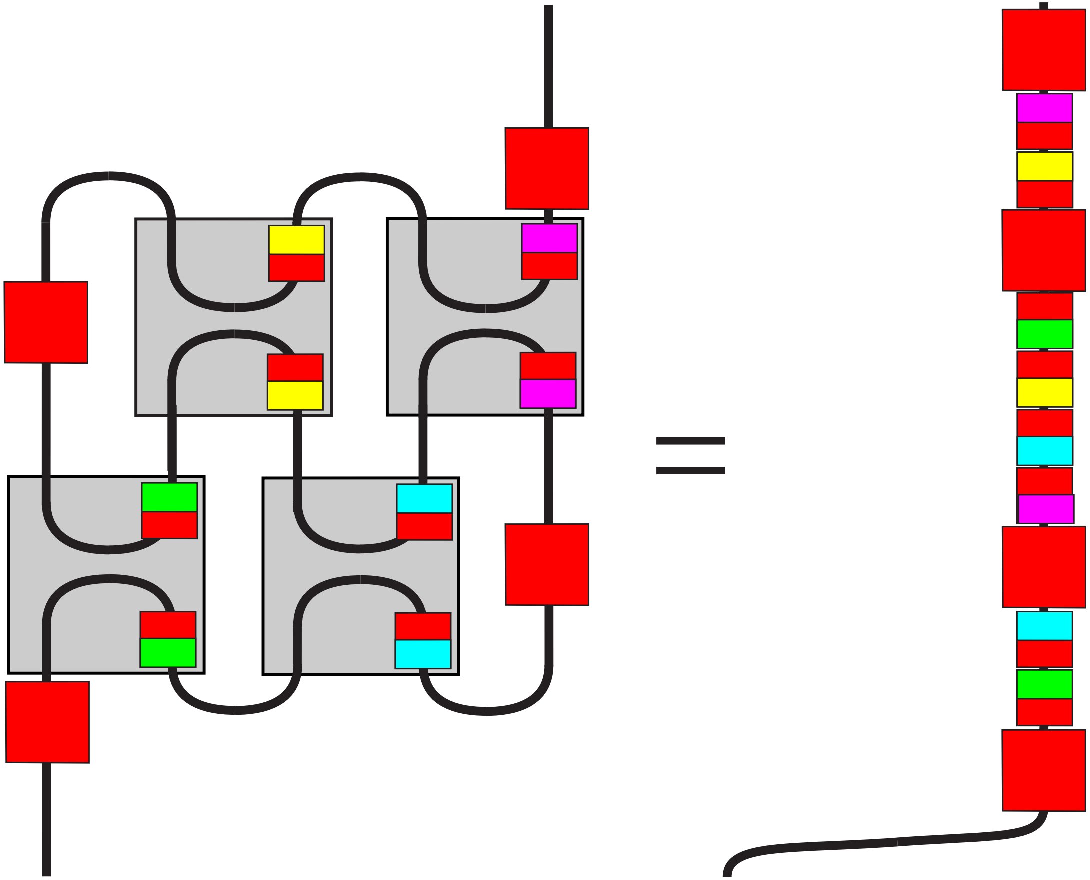




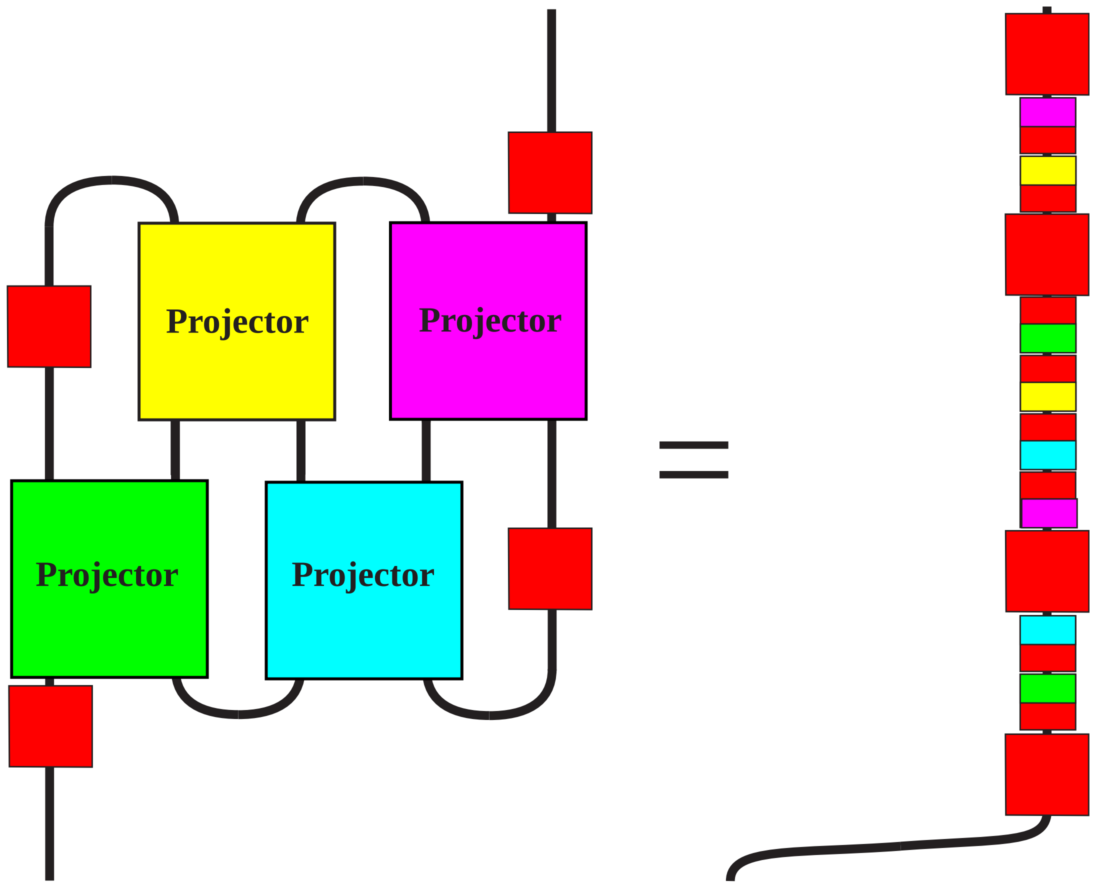
“Decorated” normalization



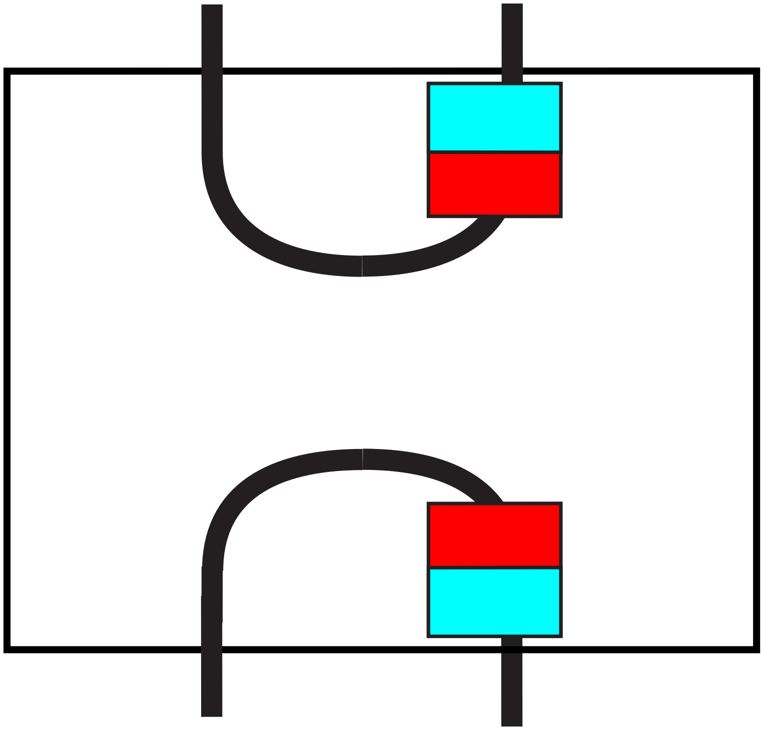
“Decorated” normalization



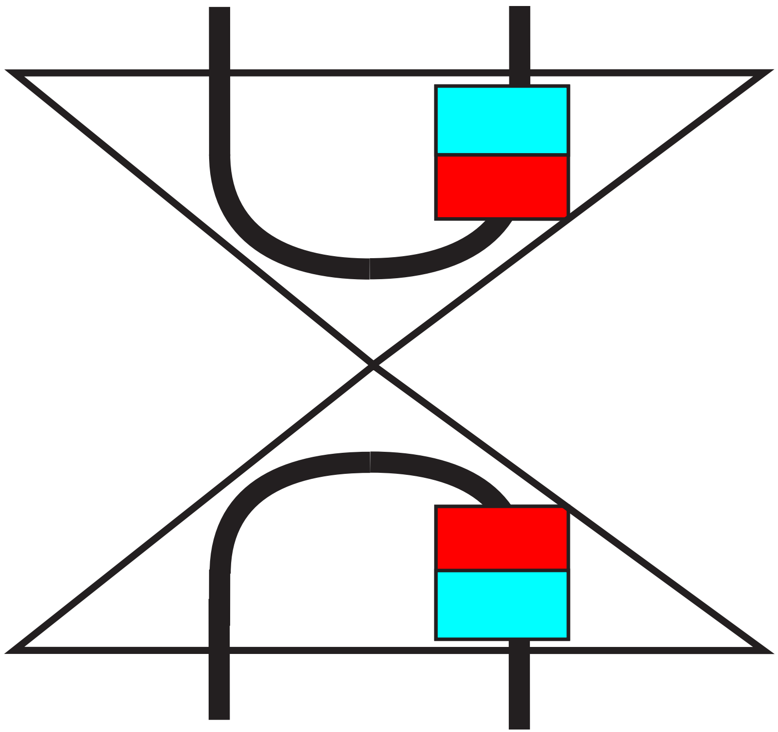
“Decorated” normalization



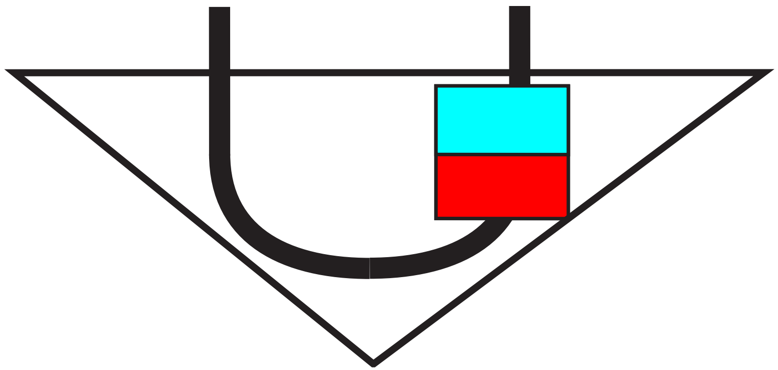
Bipartite projector



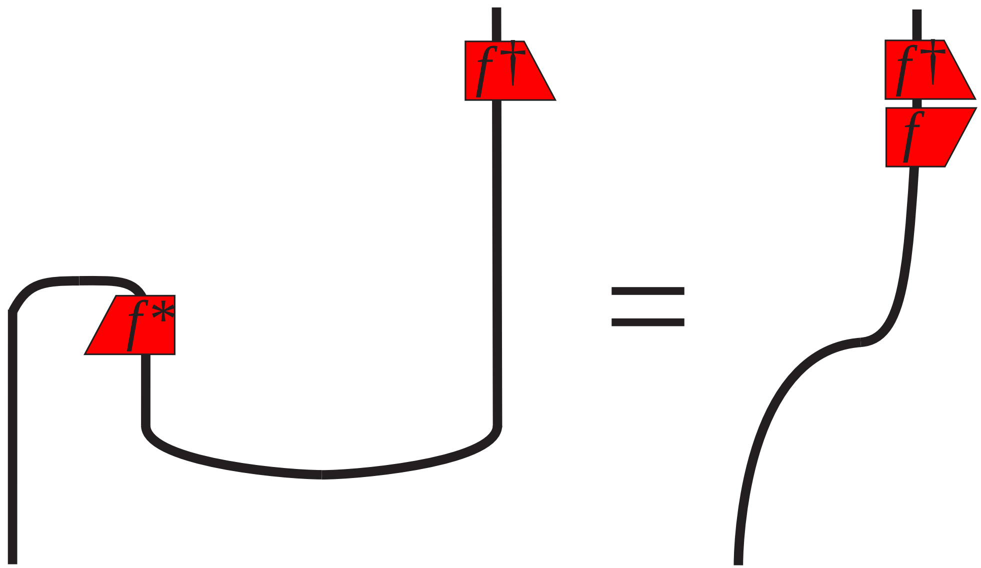
Bipartite ket & bra



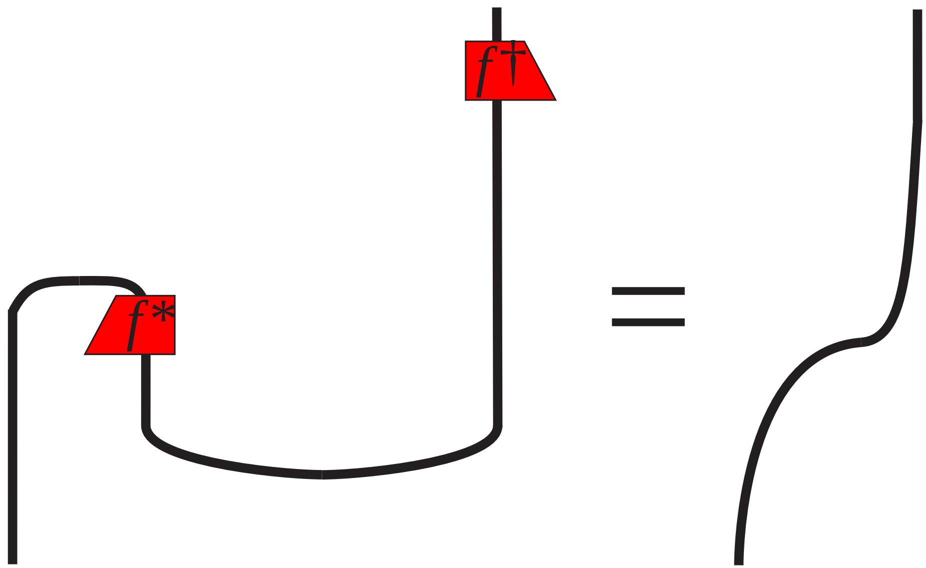
Bipartite state



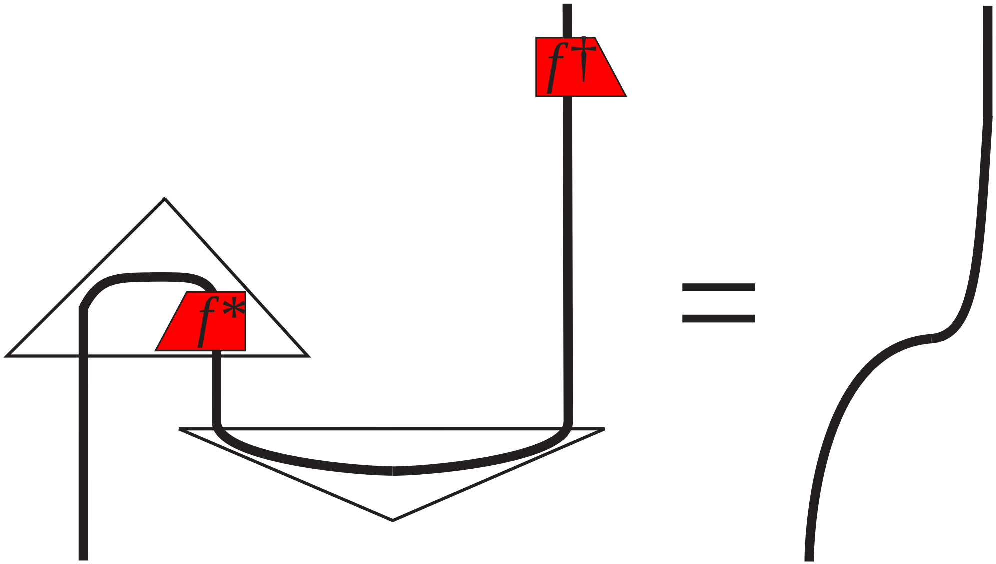
⇒ Jamiolkowski isomorphism

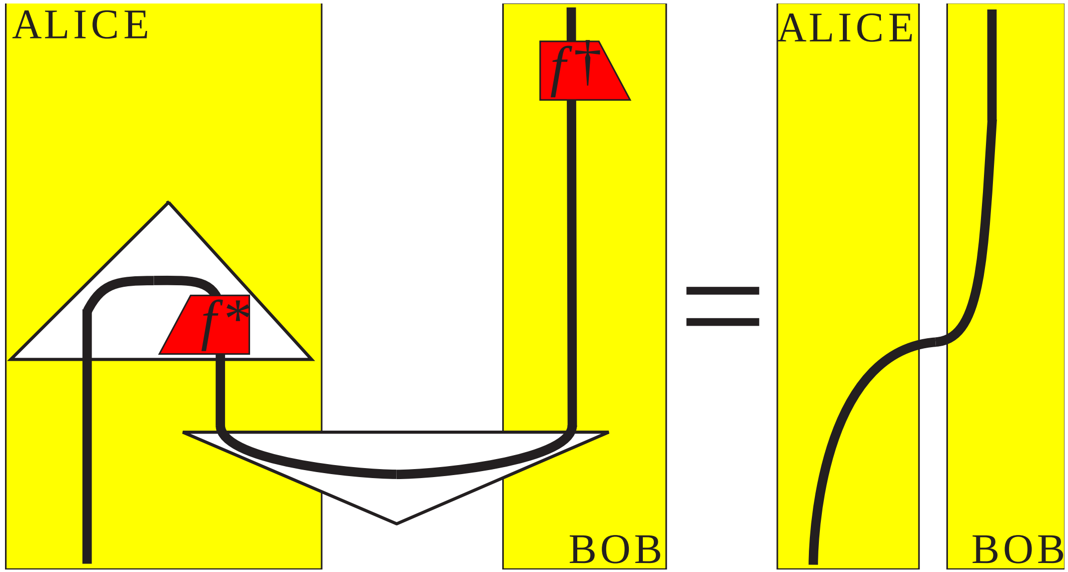


f*
f†
=
f†
f



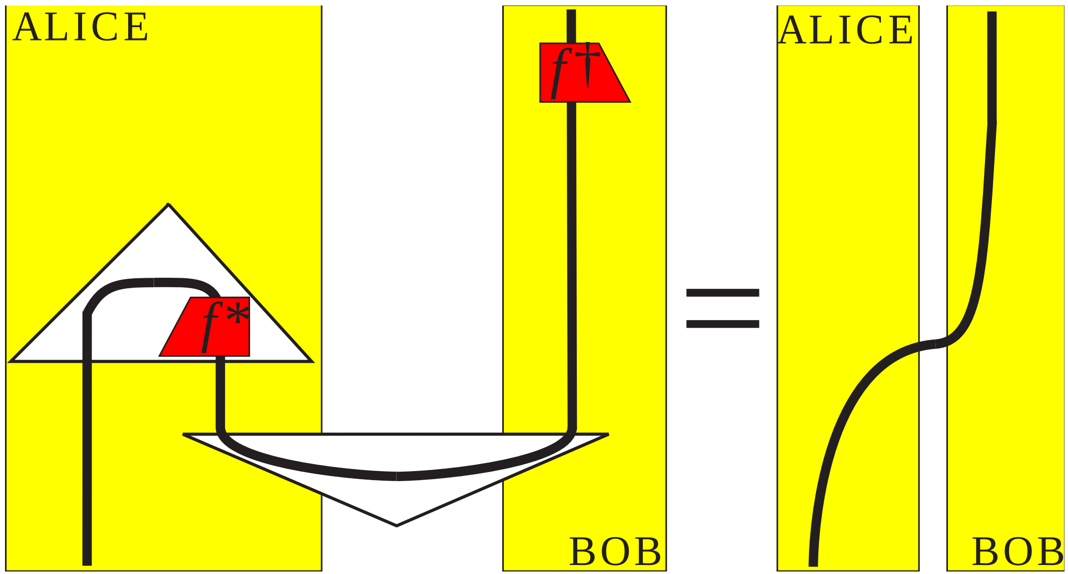
f†
f*
=


f^*
$f^\dagger$
=

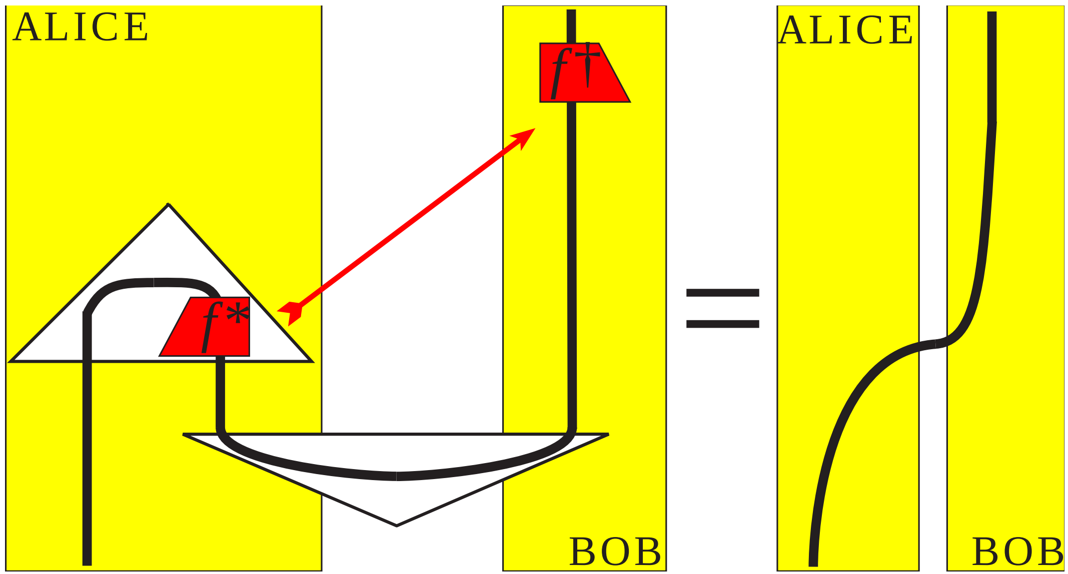


$\Rightarrow$ **Quantum teleportation**

Classical data flow?



Classical data flow?



CLASSICAL STRUCTURE

Coecke-Pavlovic (2006) quant-ph/0608035v1

Carboni-Walters (1986) *Cartesian bicategories* I.

quantum data cannot be cloned nor deleted

quantum data cannot be cloned nor deleted

classical data CAN be cloned and deleted

NON-FEATURE:

quantum data cannot be cloned nor deleted

FEATURE:

classical data CAN be cloned and deleted

NON-FEATURE:

quantum data cannot be cloned nor deleted

FEATURE:

classical data CAN be cloned and deleted

Classical data comes with cloning and deleting:

$$(X\,,\delta : X \to X \otimes X\,, \epsilon : X \to \mathrm{I})$$

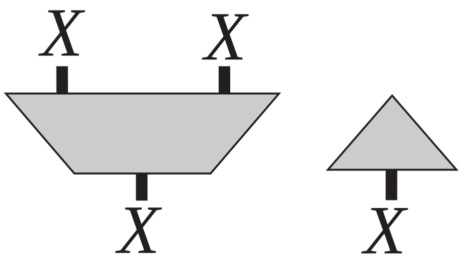
NON-FEATURE:

quantum data cannot be cloned nor deleted

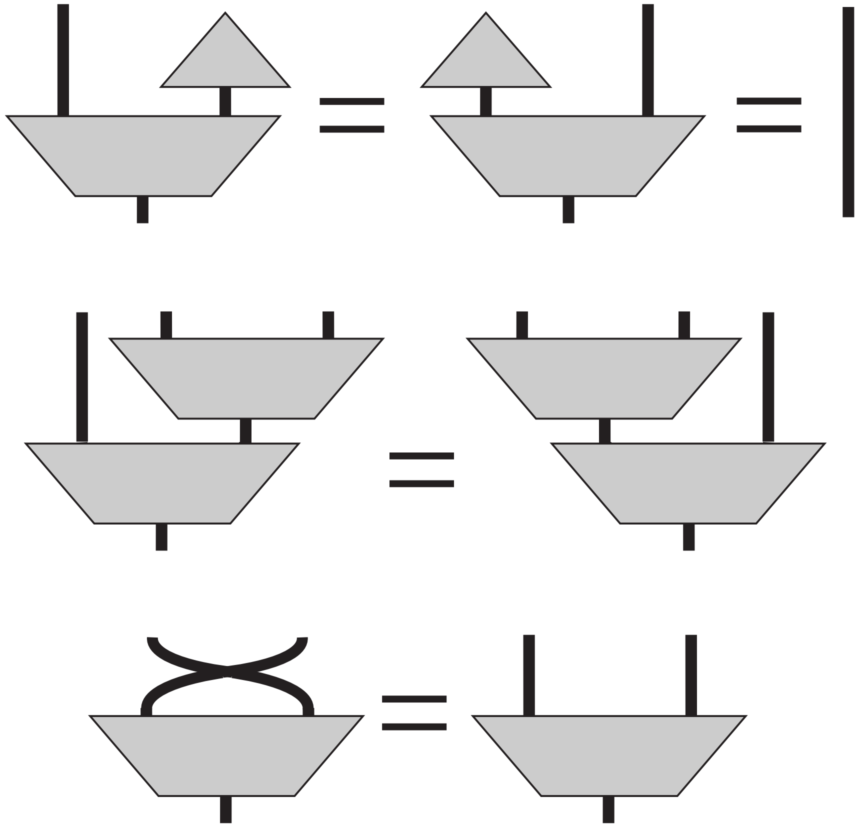
FEATURE:

classical data CAN be cloned and deleted

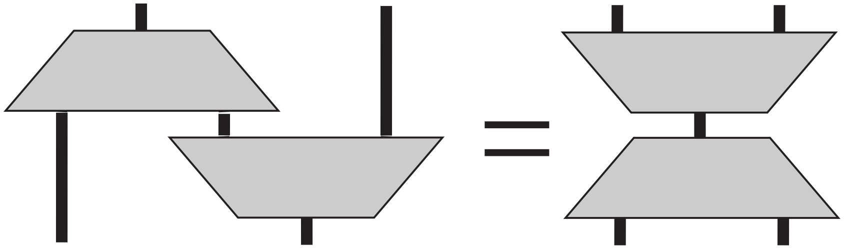
Classical data comes with cloning and deleting:



System with classical structure

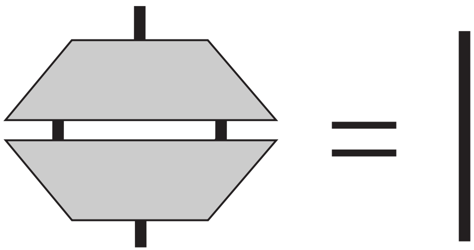


System with classical structure

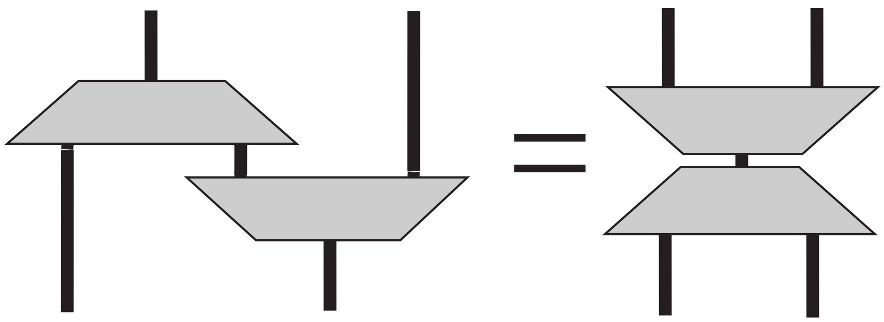


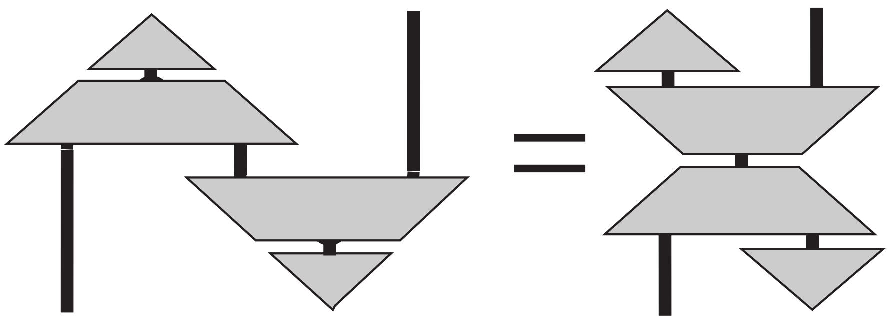
"Frobenius"

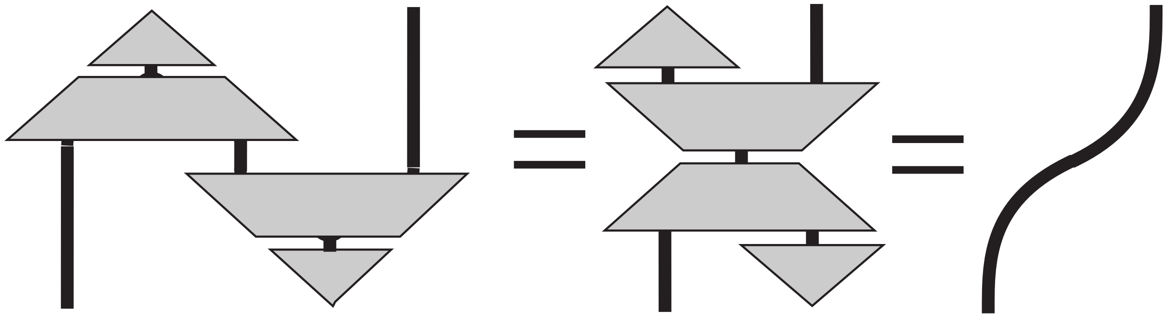
(Carboni-Walters 1987 *Cartesian bicategories* I)



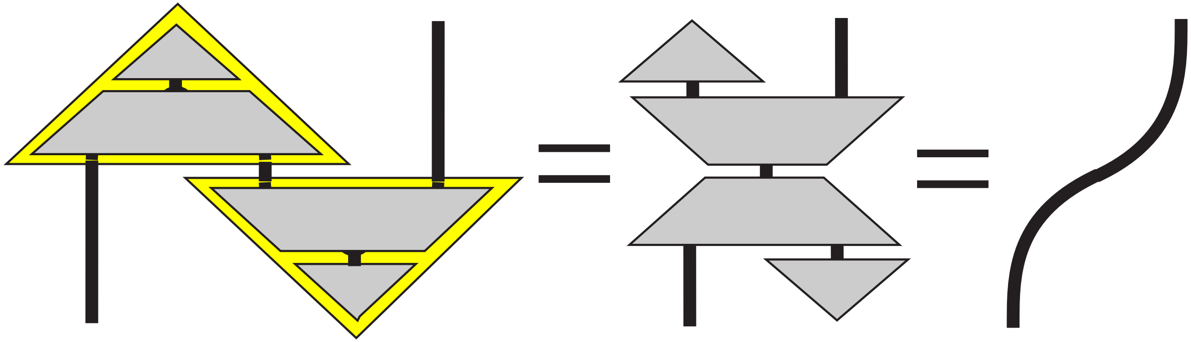
"normalisation"



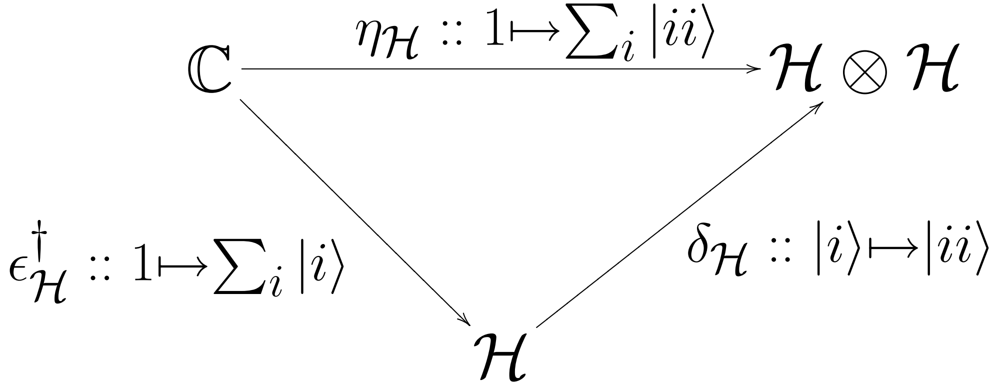




Classical structure ⇒ quantum structure



In Hilb **the Bell-state decomposes as:**



In Hilb the Bell-state decomposes as:

$$\mathbb{C} \xrightarrow{\ \eta_{\mathcal{H}} \,::\, 1 \mapsto \sum_i |ii\rangle\ } \mathcal{H} \otimes \mathcal{H}$$

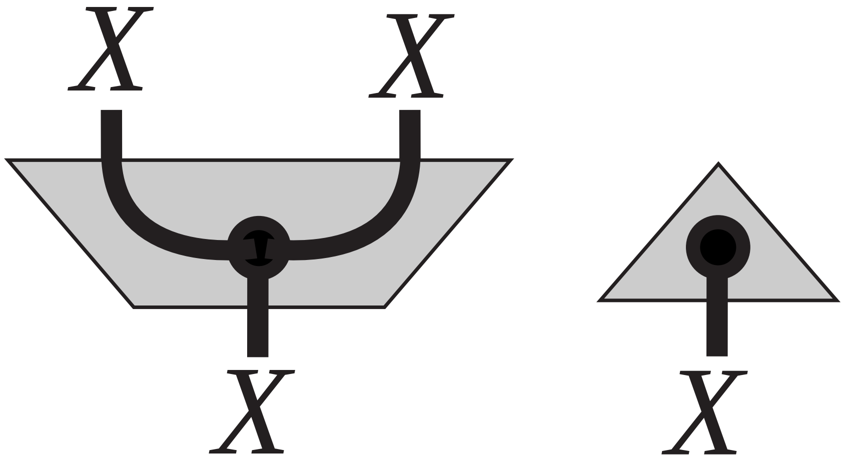
$$\mathbb{C} \xrightarrow{\ \epsilon^{\dagger}_{\mathcal{H}} \,::\, 1 \mapsto \sum_i |i\rangle\ } \mathcal{H}$$

$$\mathcal{H} \xrightarrow{\ \delta_{\mathcal{H}} \,::\, |i\rangle \mapsto |ii\rangle\ } \mathcal{H} \otimes \mathcal{H}$$

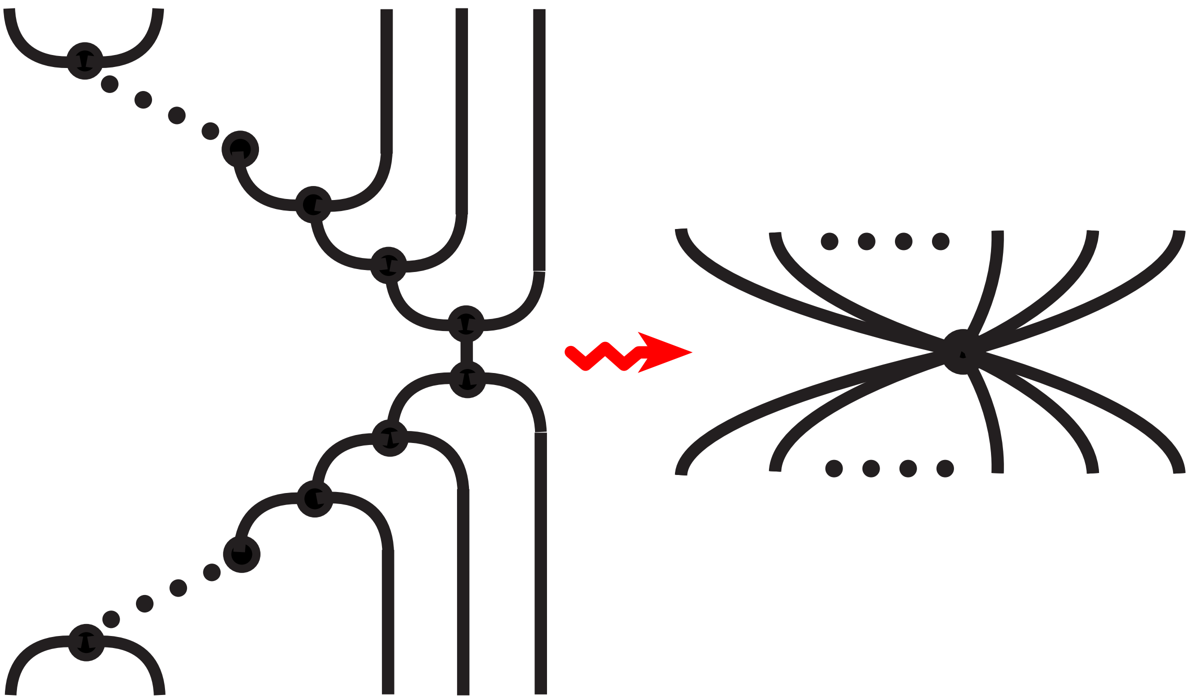
This "refinement" specifies a base!

“What’s inside the box?”

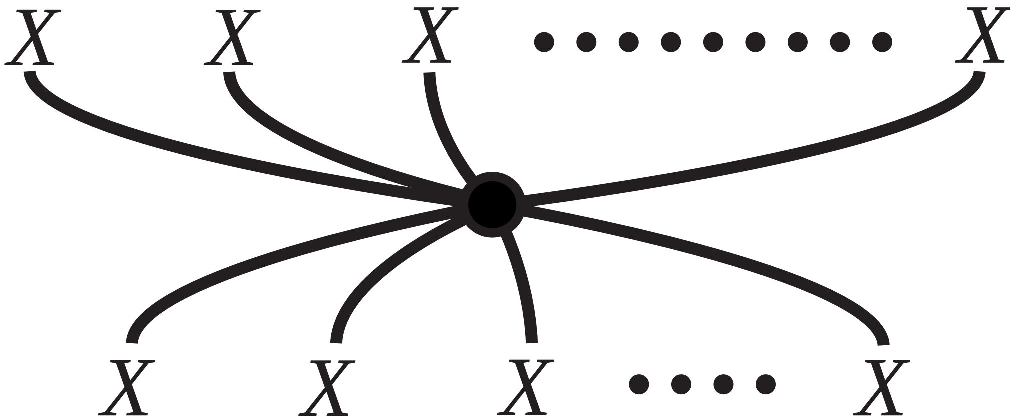
“What’s inside the box?”



Notational convention:



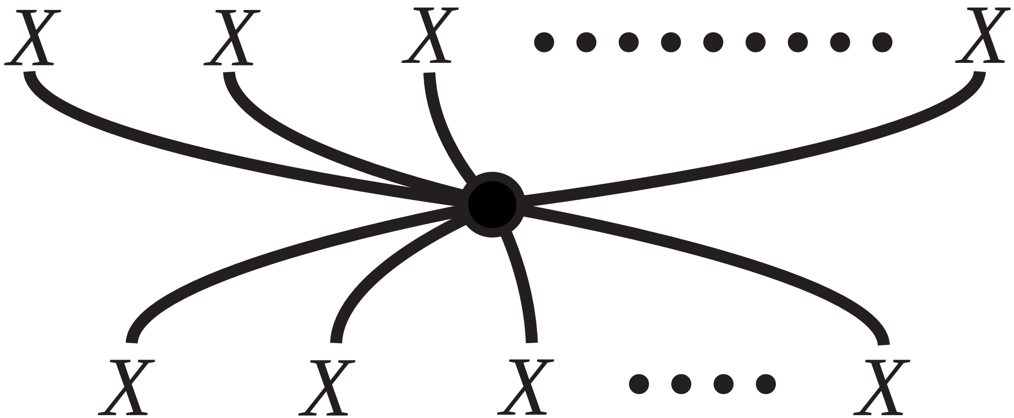
Normalisation theorem: A "connected" network build from δ, $\delta^\dagger$, ϵ, $\epsilon^\dagger$ admits a 'spider-like' normal form:



Kock, J. (2003) ***Frobenius algebras and 2D TQFTs*****.**

Coecke-Paquette (2006) ***POVMs & Naimark's thm without sums*****.**

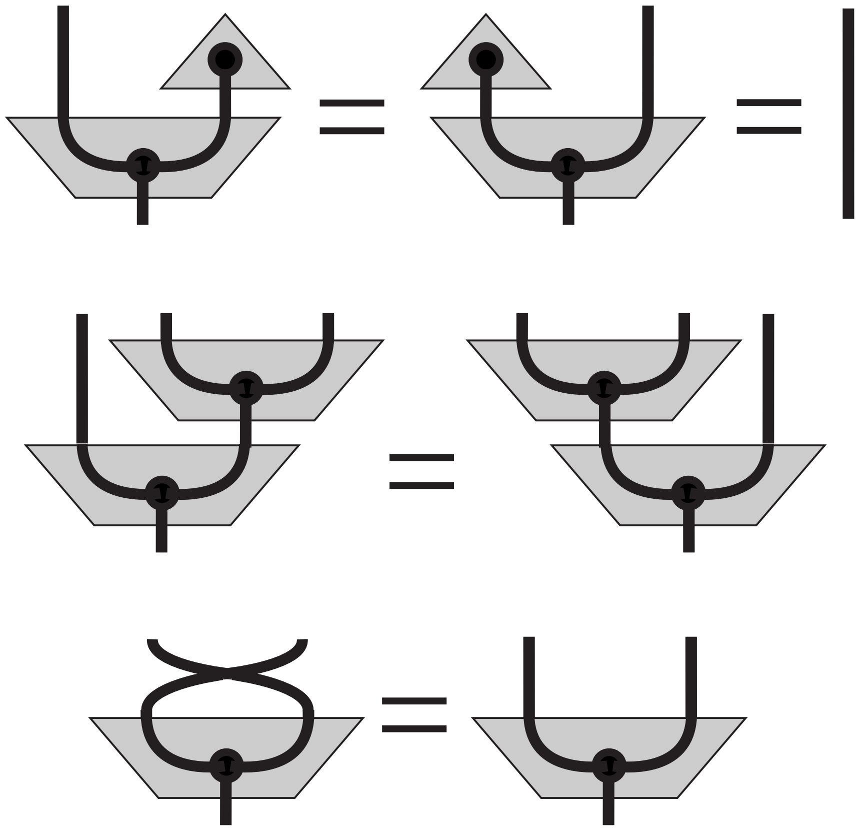
Normalisation theorem: A "connected" network build from δ, $\delta^\dagger$, ϵ, $\epsilon^\dagger$ admits a 'spider-like' normal form:

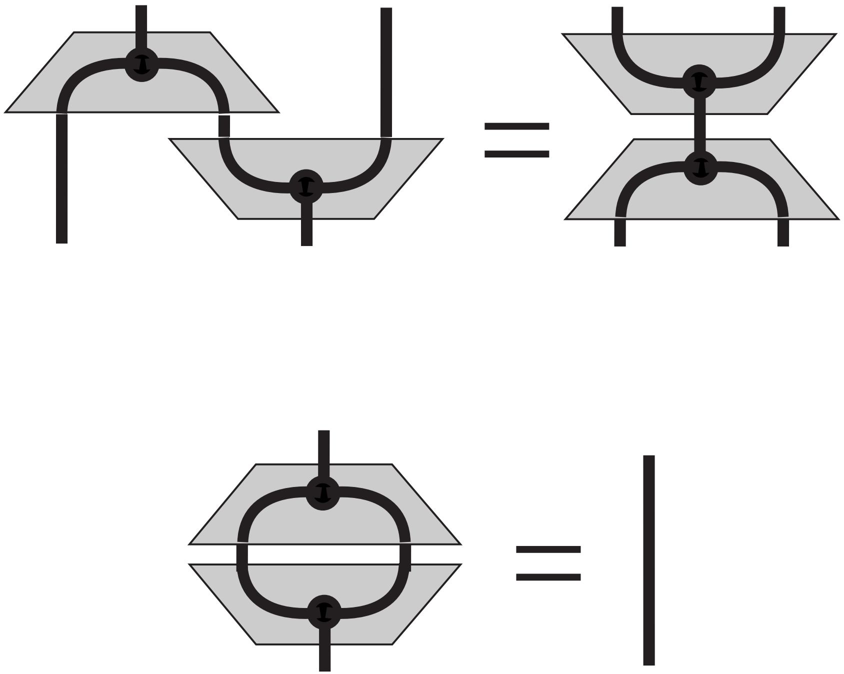


proof $\sim$ "fusion" of dots $\Rightarrow$ graphical rewrite system

Kock, J. (2003) ***Frobenius algebras and 2D TQFTs*****.**

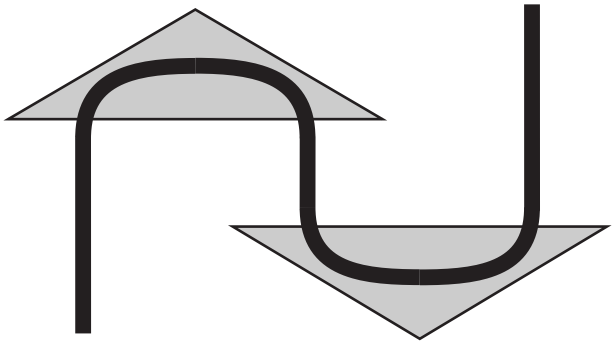
Coecke-Paquette (2006) ***POVMs & Naimark's thm without sums*****.**



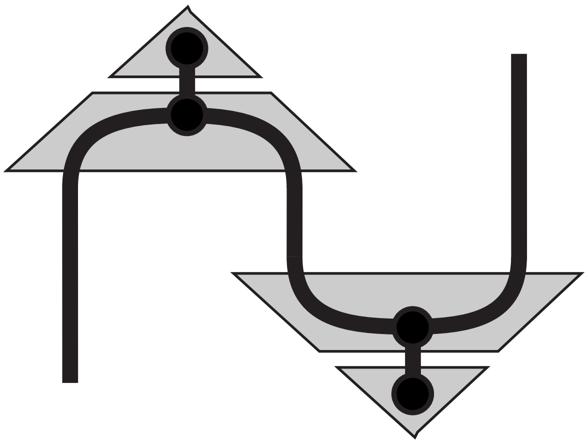


All five axioms follow from spider-normal-form.

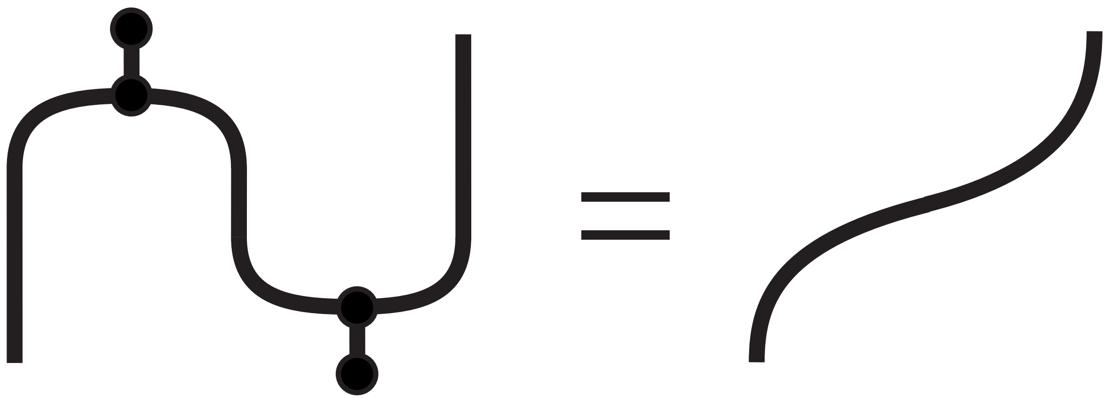
Summary: refining quantum structure



Summary: refining quantum structure



Summary: refining quantum structure



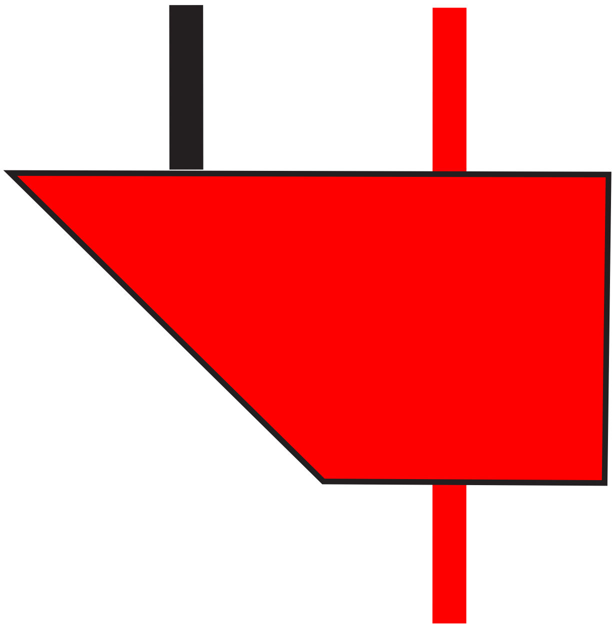
QUANTUM-CLASSICAL FLOW

Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$

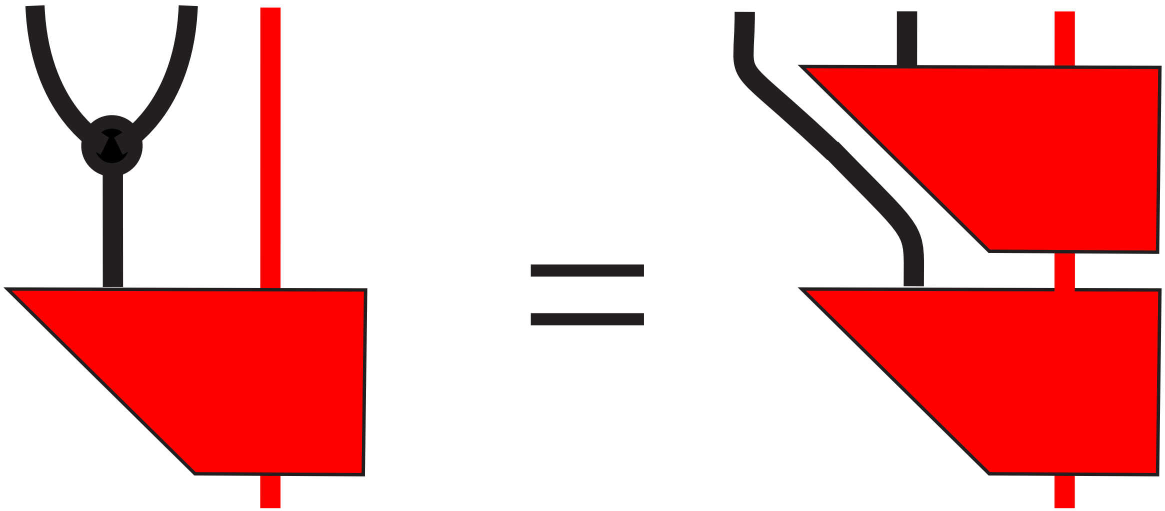
Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$



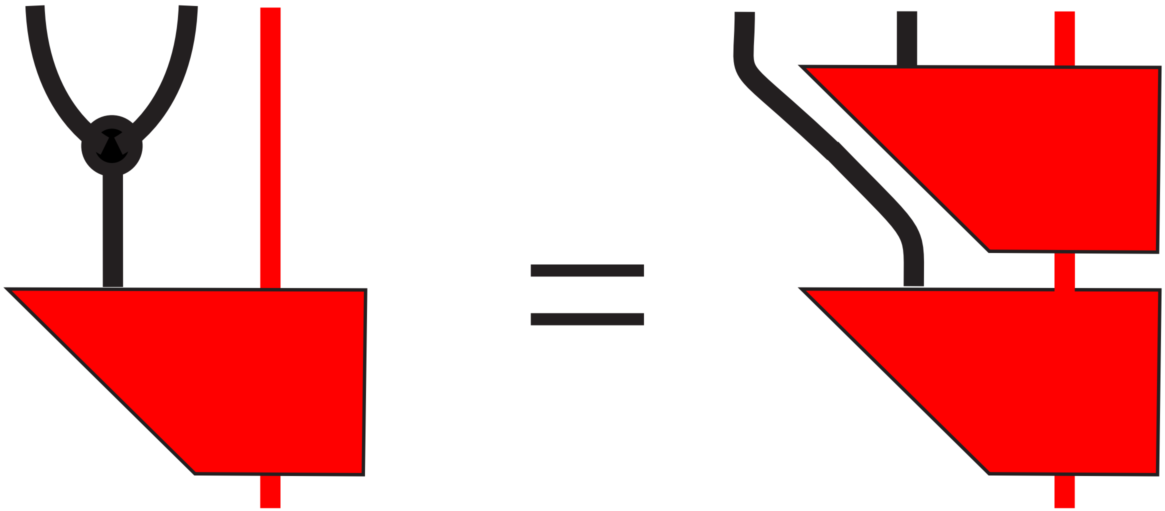
Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$



Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$



$\Rightarrow$ **von Neumann projection postulate.**

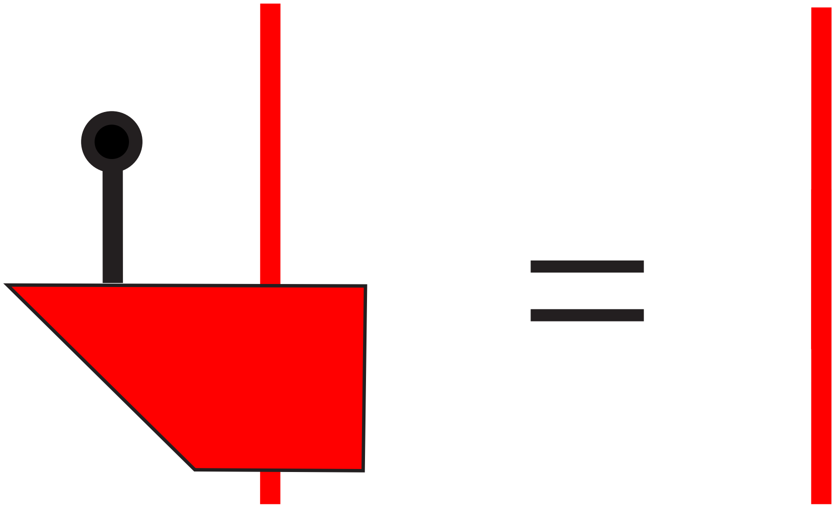
Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$

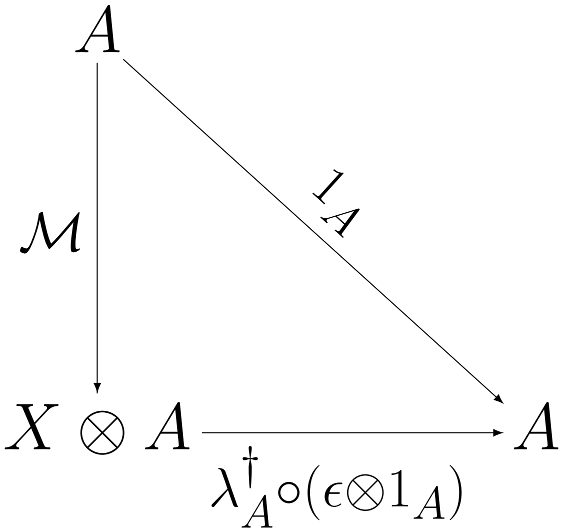
$$\begin{array}{ccc}
A & \xrightarrow{\quad \mathcal{M} \quad} & X \otimes A \\
\Big\downarrow{\scriptstyle \mathcal{M}} & & \Big\downarrow{\scriptstyle 1_X \otimes \mathcal{M}} \\
X \otimes A & \xrightarrow[\delta \otimes 1_A]{} & X \otimes X \otimes A
\end{array}$$

Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$

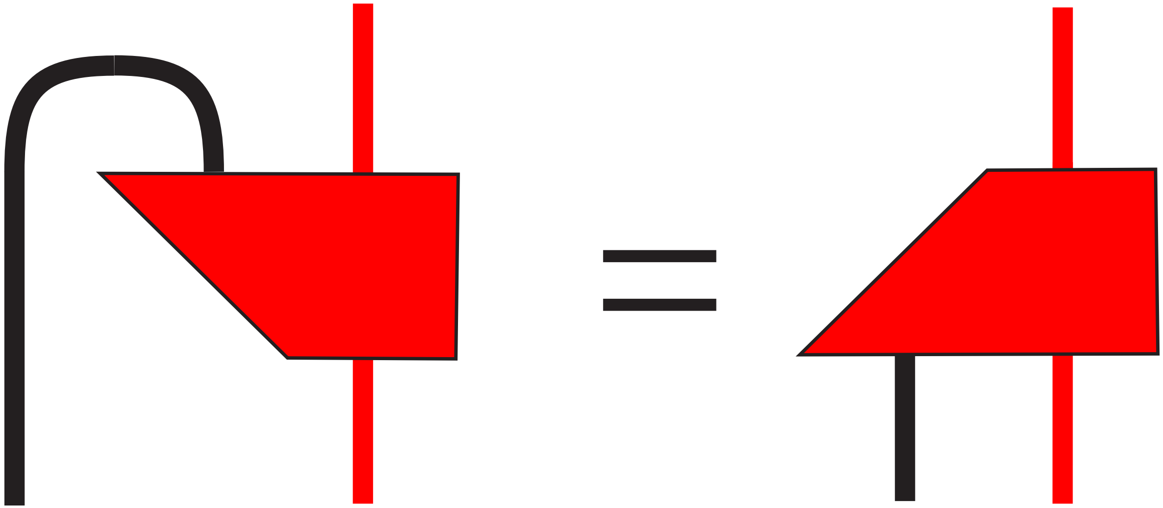


Quantum measurement:



Quantum measurement:

$$\mathcal{M} : A \to X \otimes A$$



$\Rightarrow$ **"indexed" self-adjointness.**

Thm. **Self-adjoint Eilenberg-Moore coalgebras for**

$$\mathcal{H} \otimes - : \mathbf{FdHilb} \to \mathbf{FdHilb}$$

are exactly $\mathsf{dim}\mathcal{H}$**-outcome quantum measurements.**

Thm. **Self-adjoint Eilenberg-Moore coalgebras for**

$$\mathcal{H} \otimes - : \mathbf{FdHilb} \to \mathbf{FdHilb}$$

are exactly $\mathsf{dim}\mathcal{H}$**-outcome quantum measurements.**

Coalg-square $\Rightarrow$

idempotence	$\mathrm{P}_i^2 = \mathrm{P}_i$
mutual orthogonality	$\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$

Coalg-triangle $\Rightarrow$

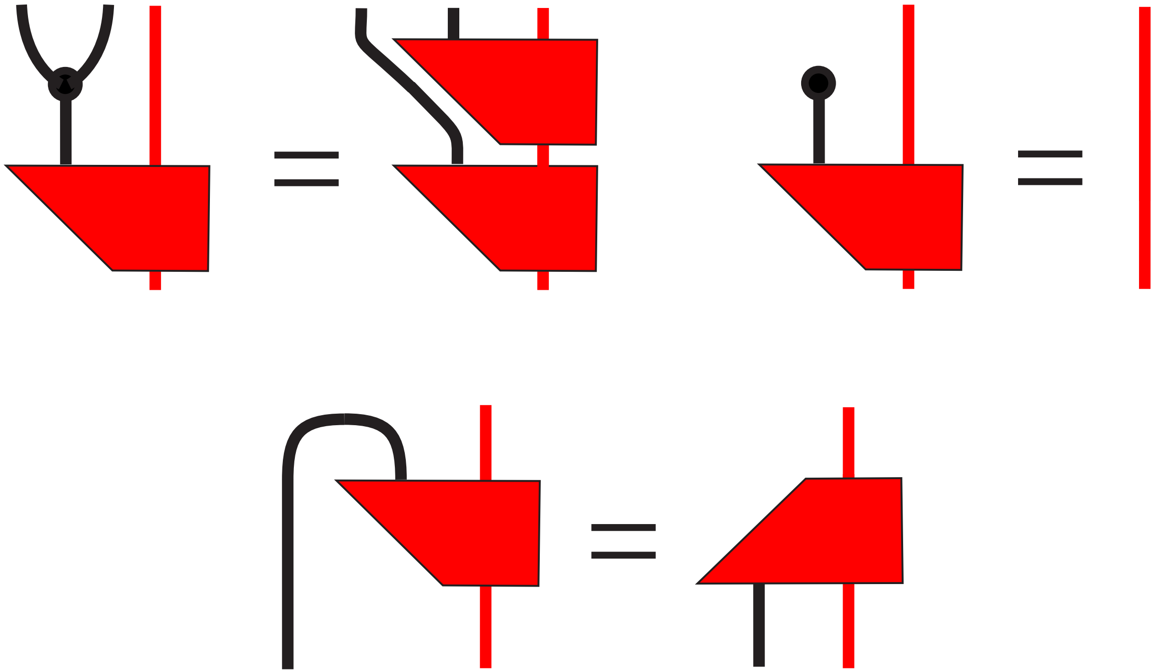
Completeness of spectrum	$\sum_i \mathrm{P}_i = 1_{\mathcal{H}}$

Self-adjointness $\Rightarrow$

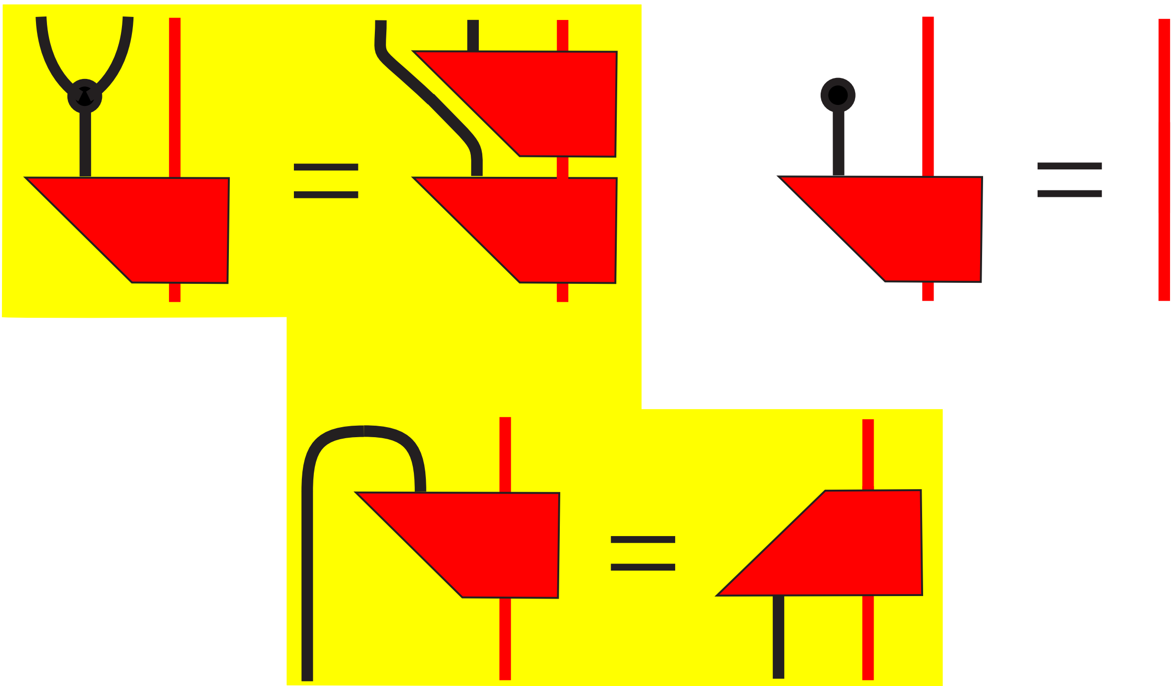
Orthogonality of projectors	$\mathrm{P}_i^\dagger = \mathrm{P}_i$

PROJECTOR SPECTRUM

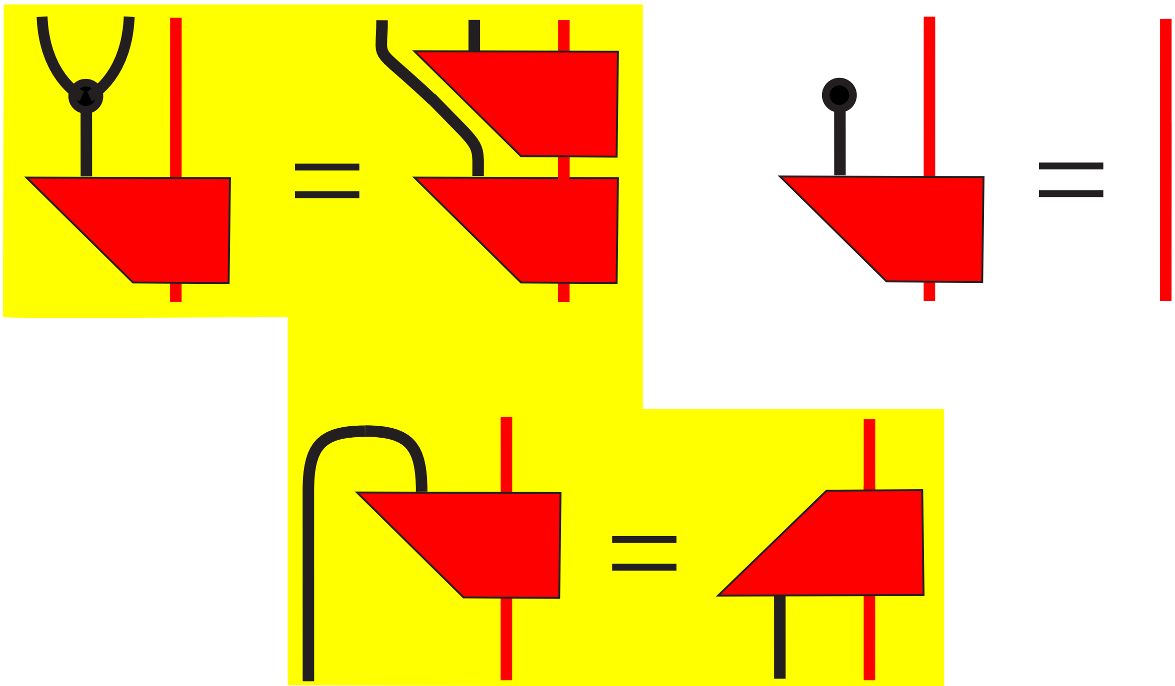
What do these mean?



What do these mean?

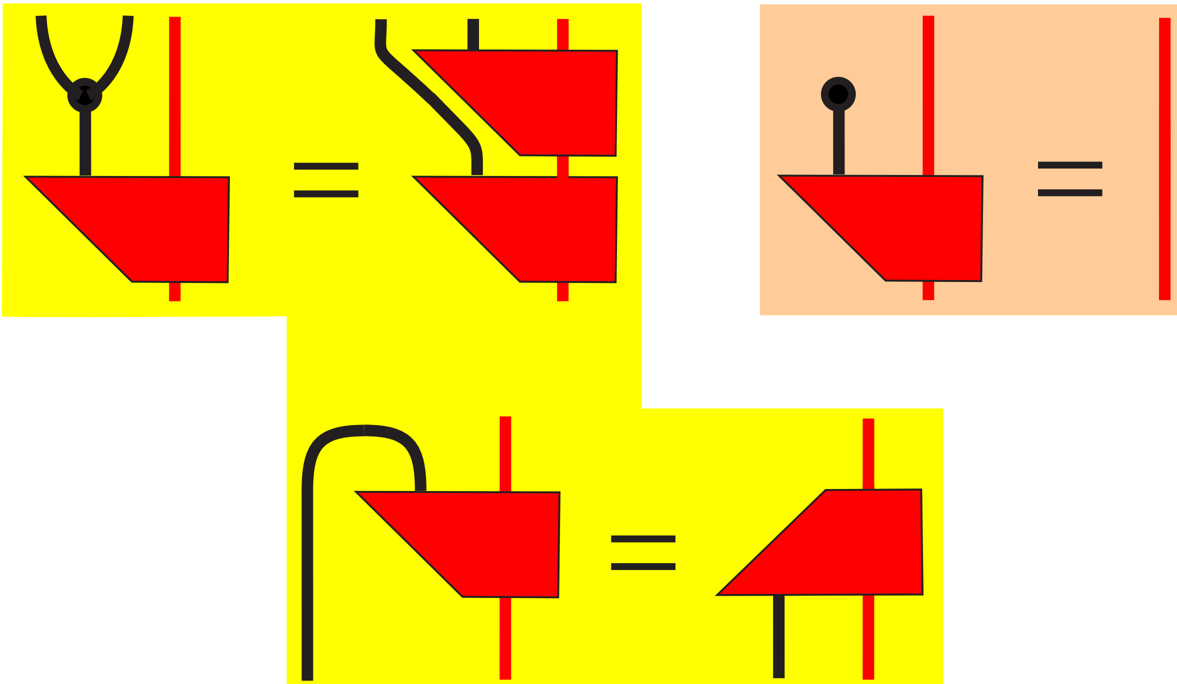


What do these mean?

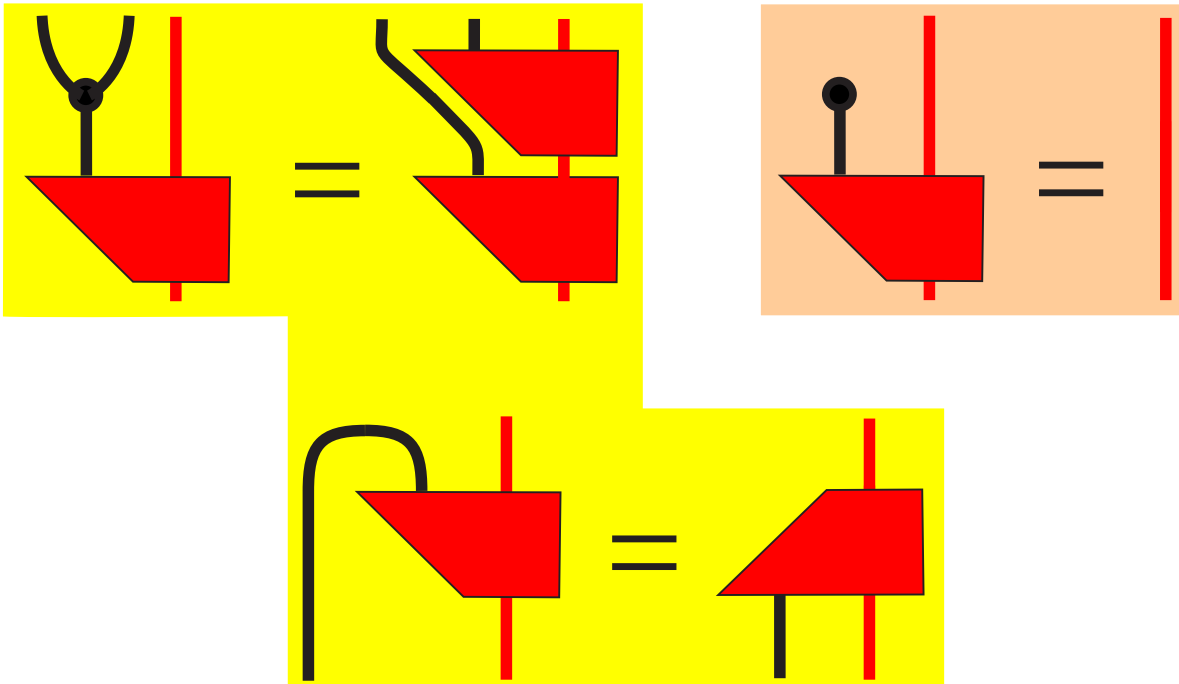


Minimal requirements for reasonable notion of measurement

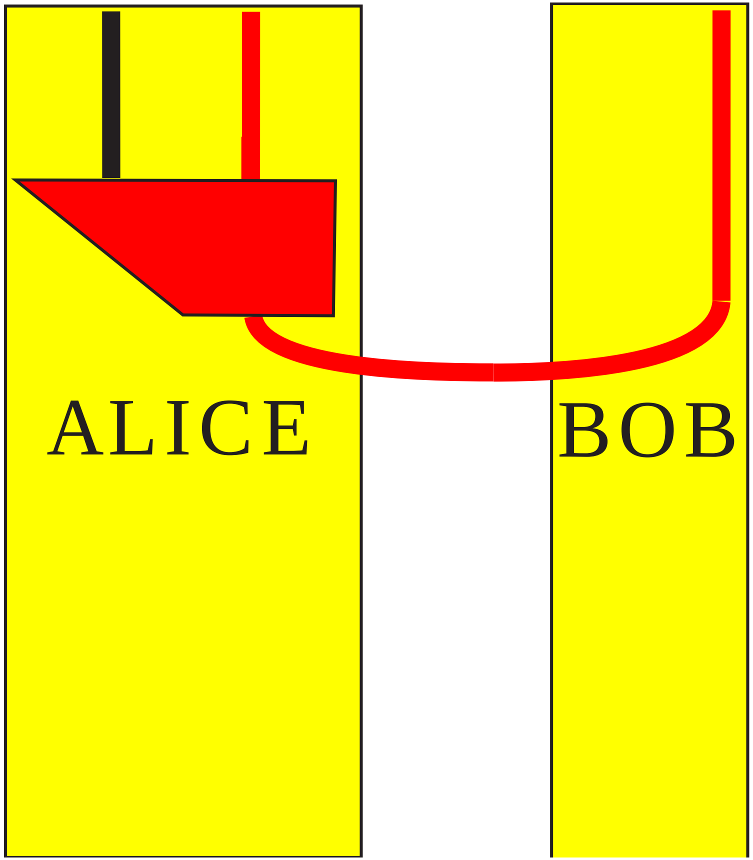
What do these mean?

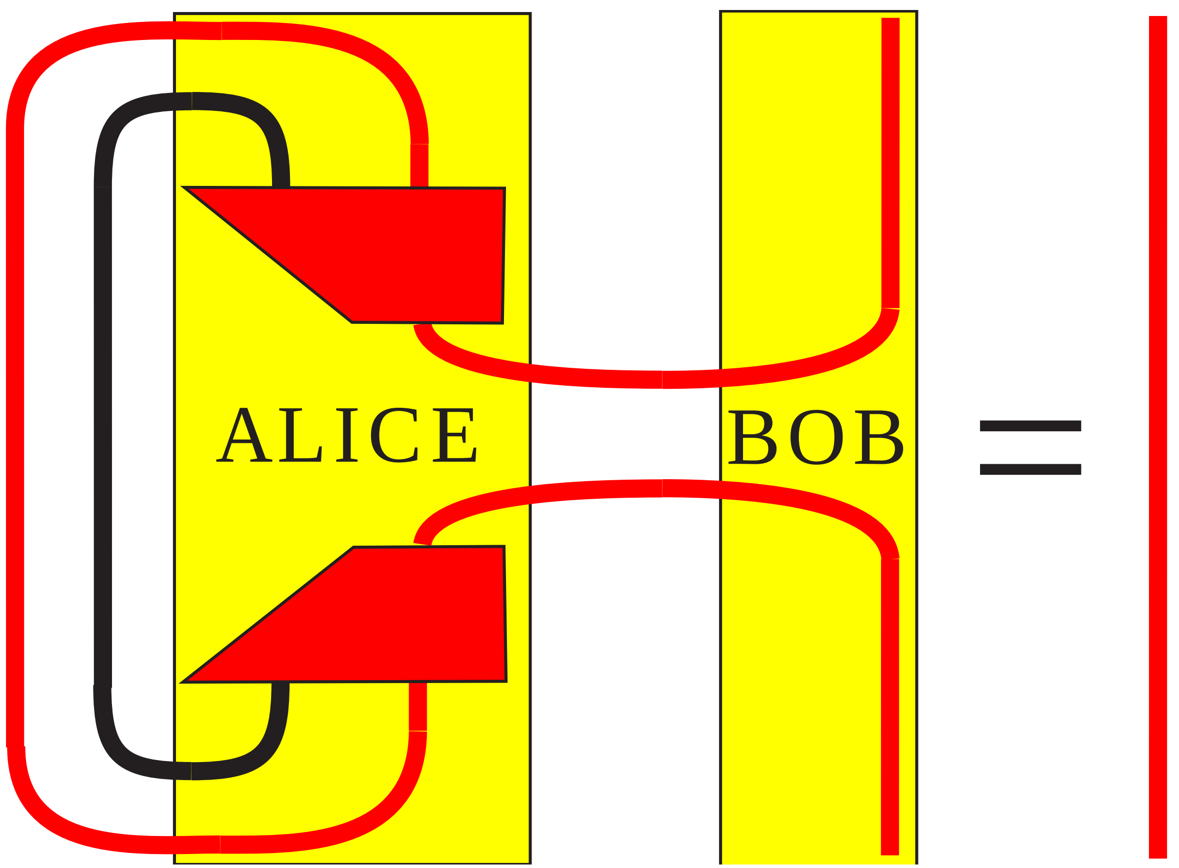


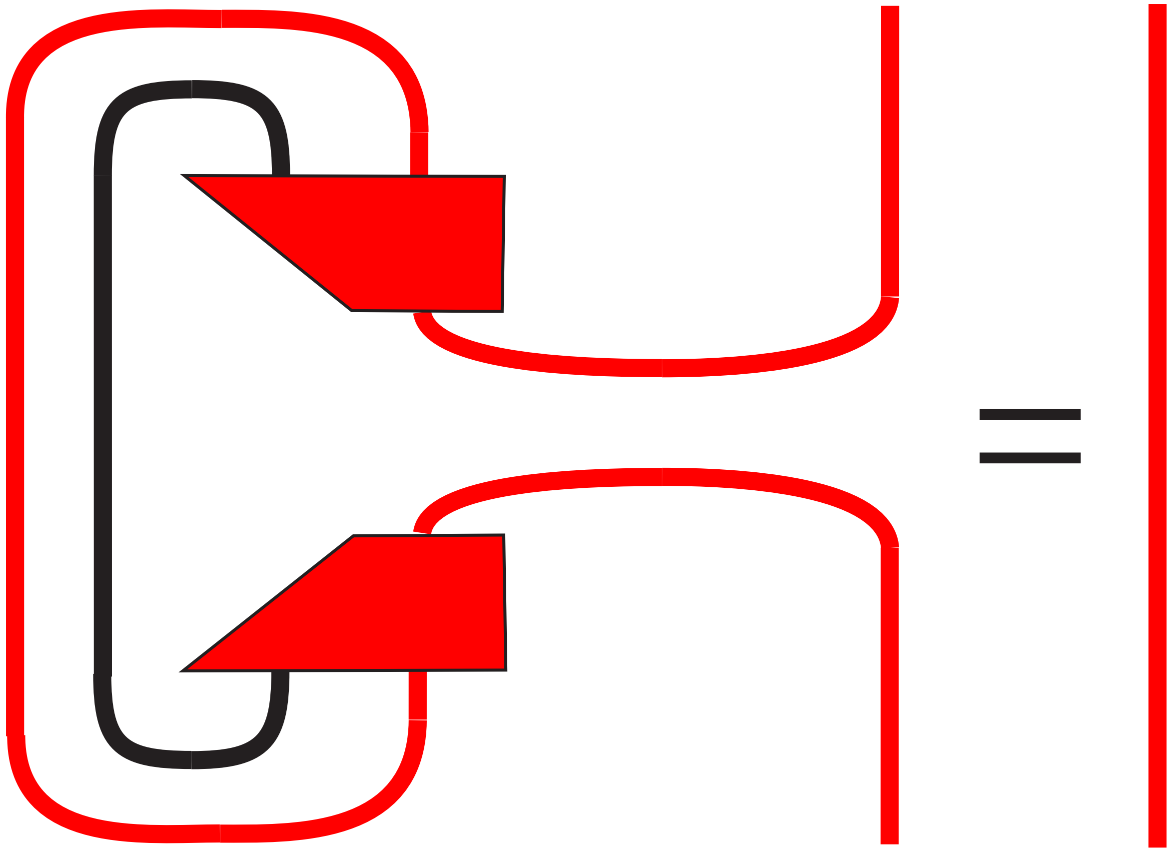
What do these mean?

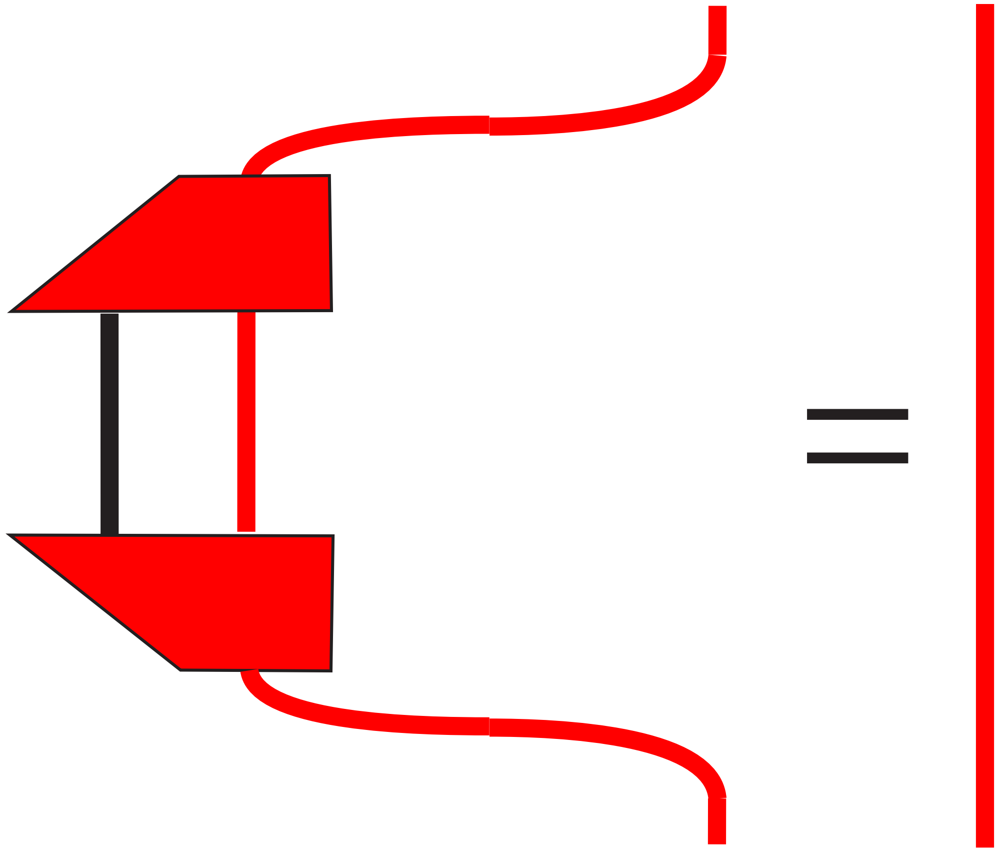


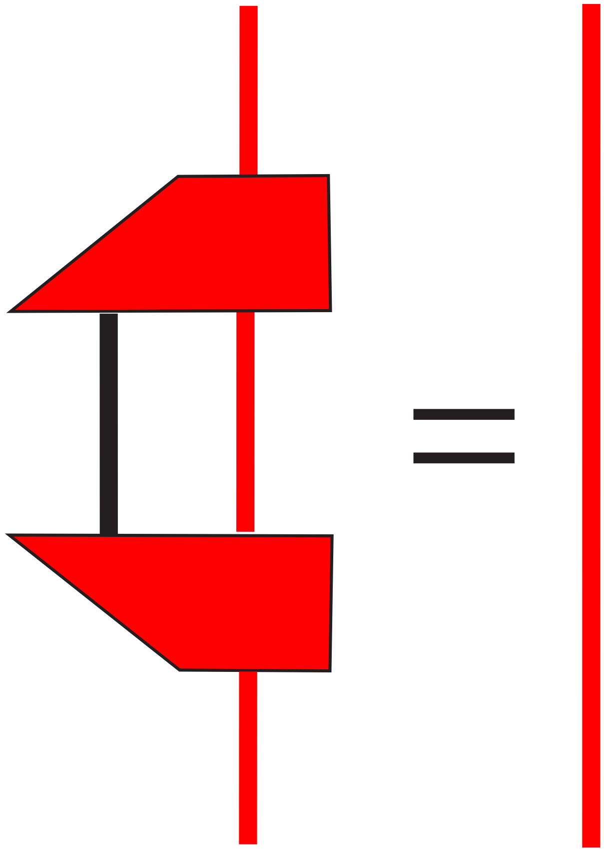
Asserts ***no-faster-than-light*** **communication**

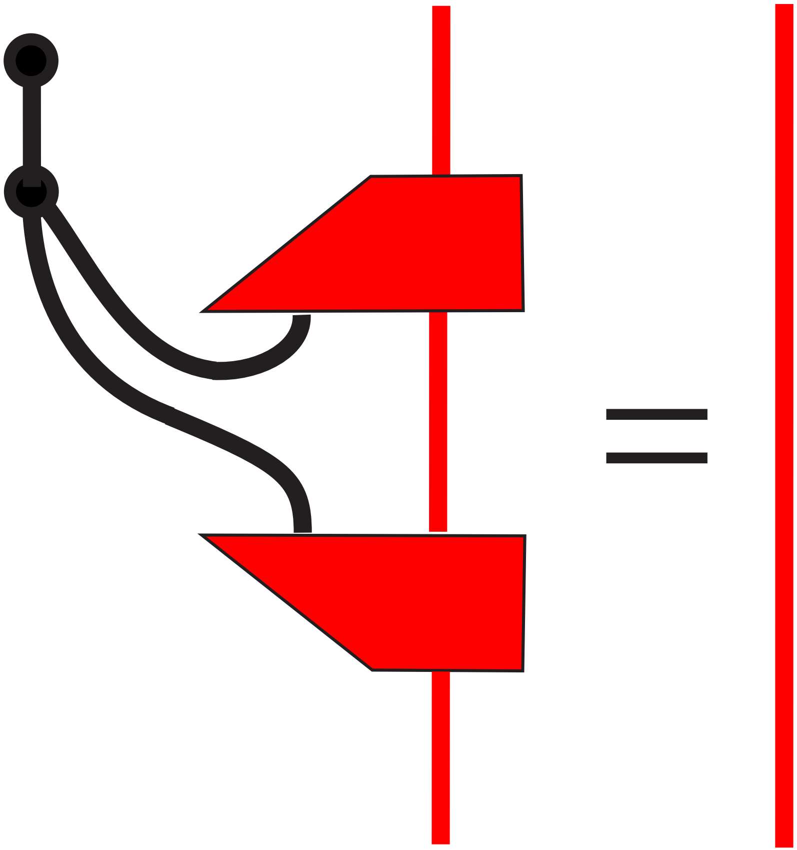

ALICE
BOB

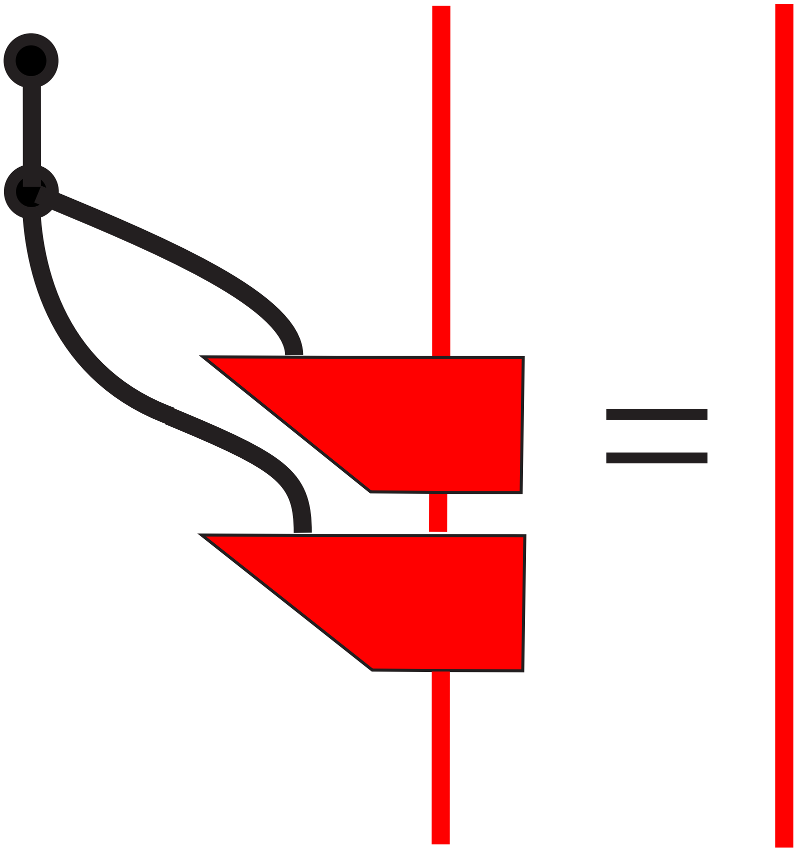

ALICE
BOB
=

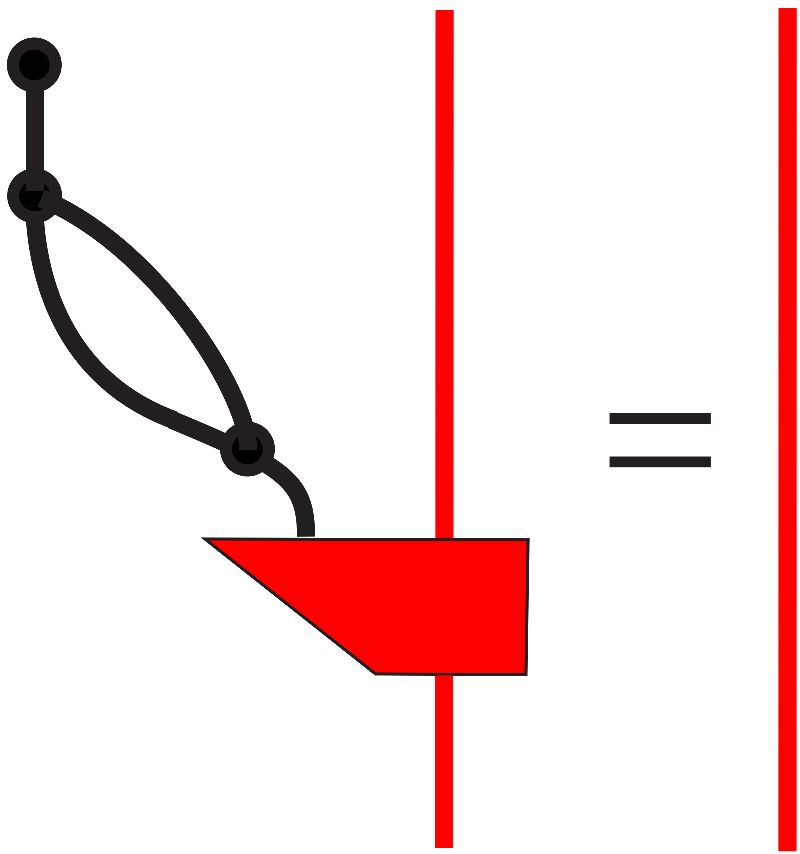


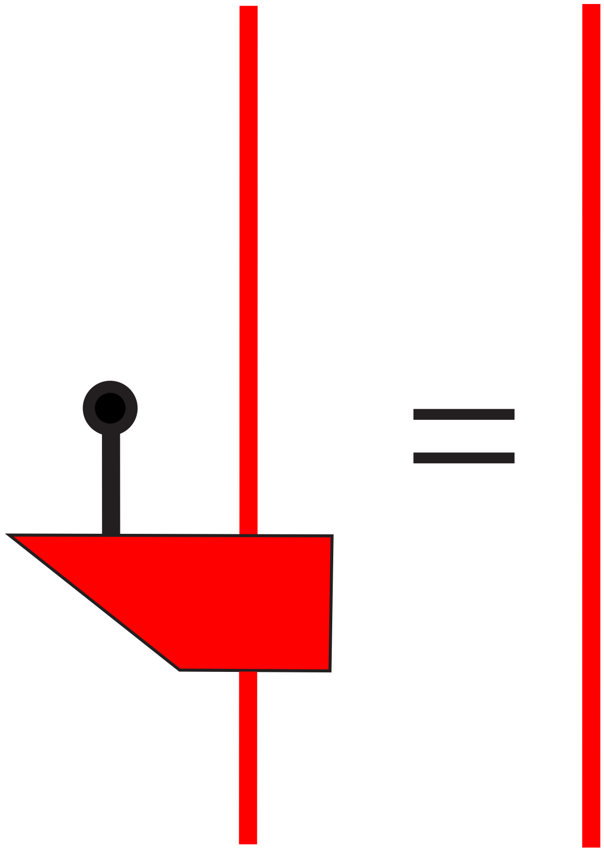




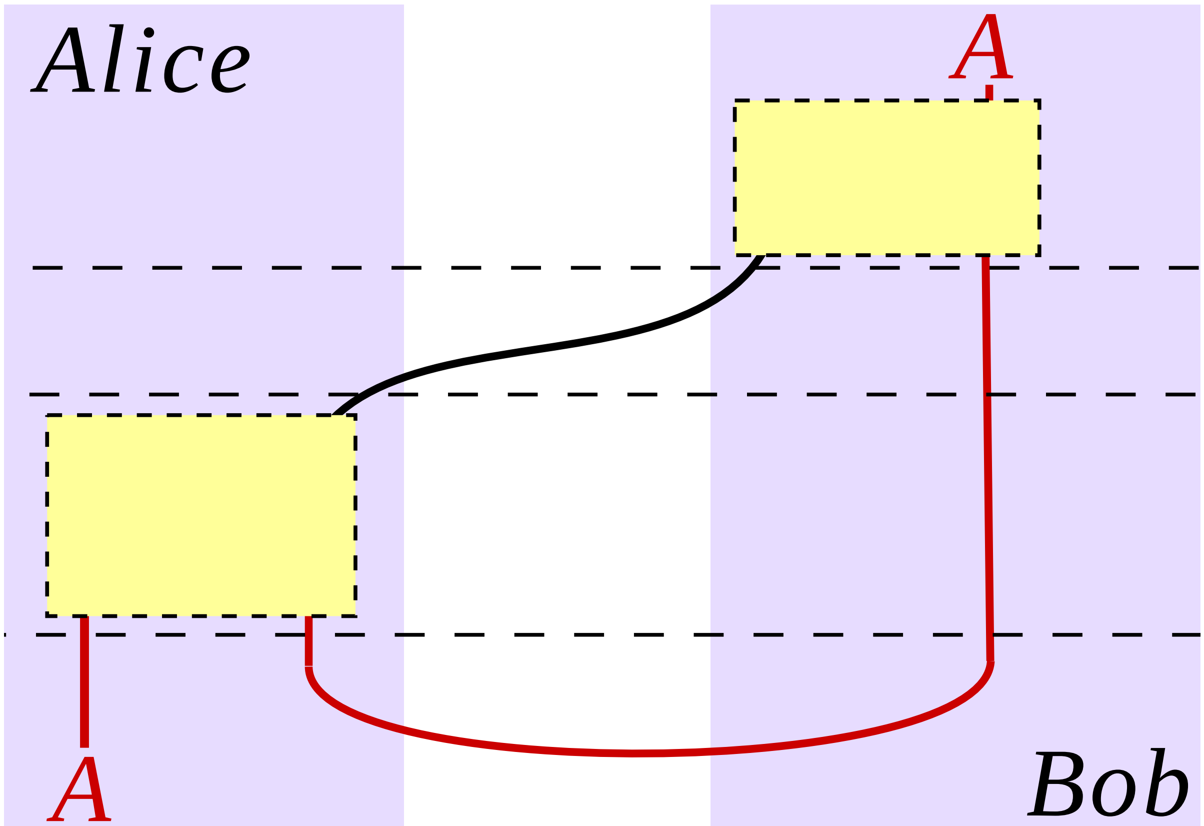




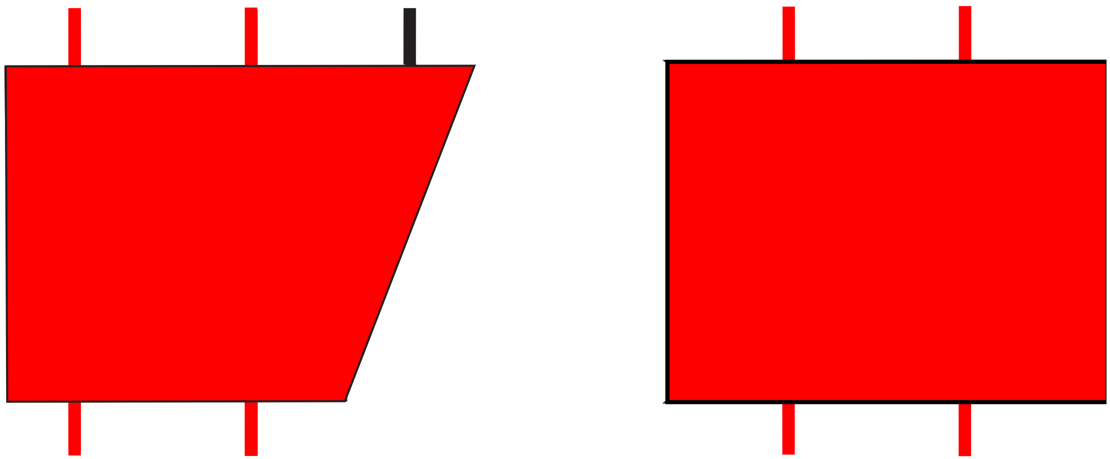




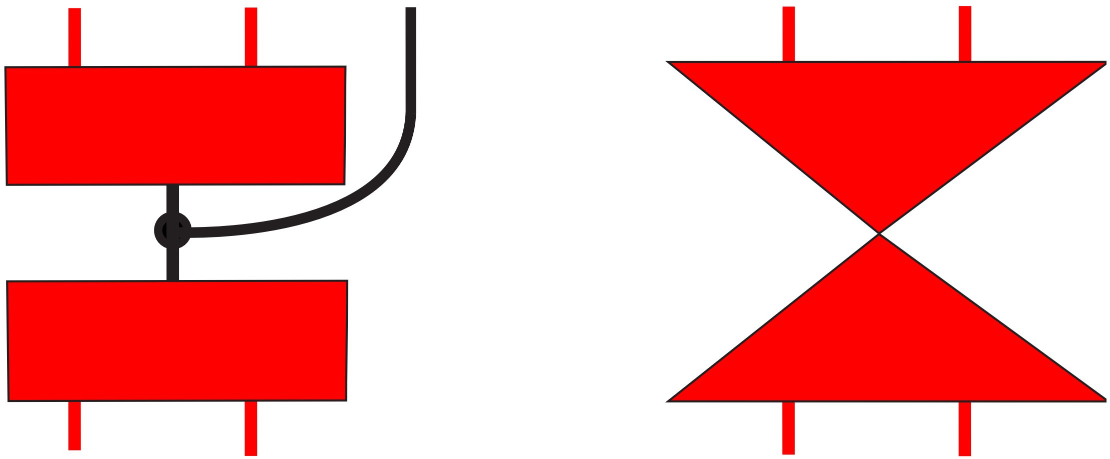
Teleportation:



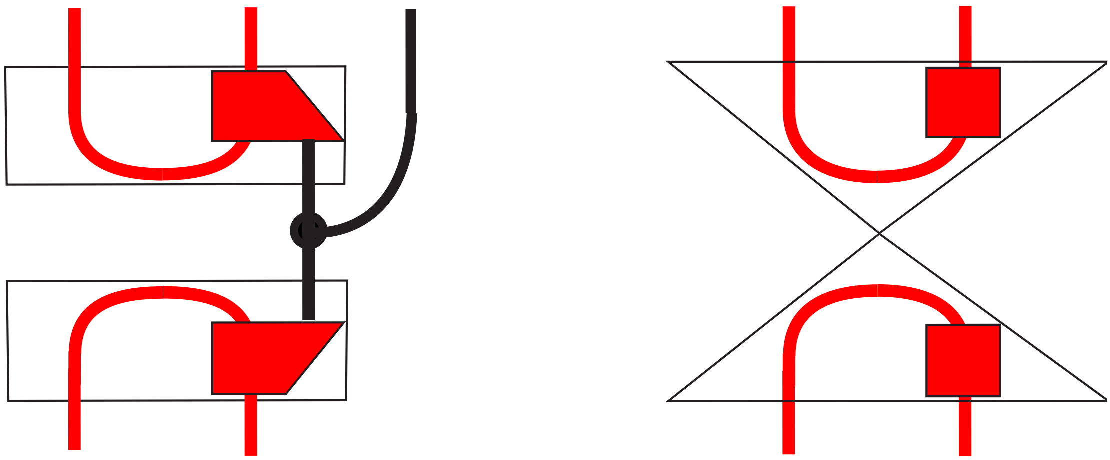
Bipartite quantum measurement:



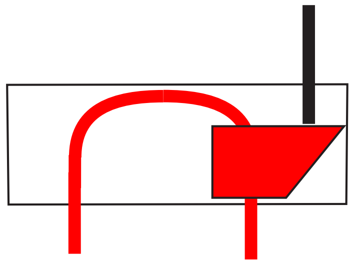
Bipartite quantum measurement:

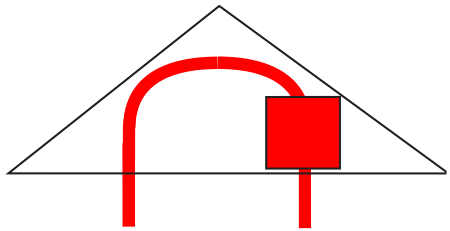


Bipartite quantum measurement:

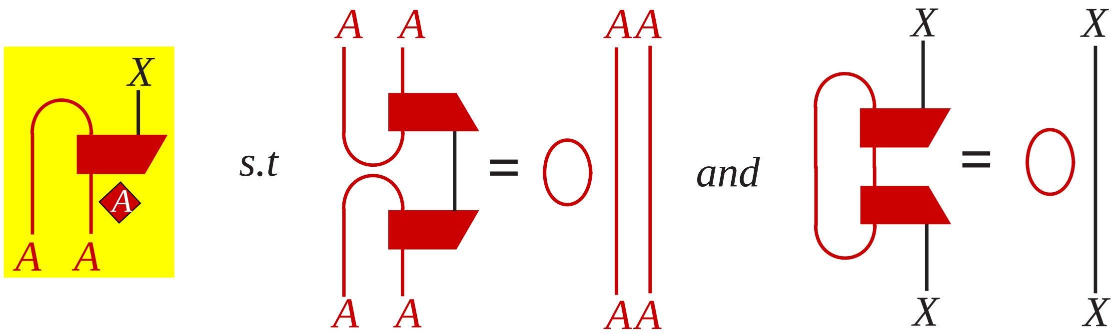


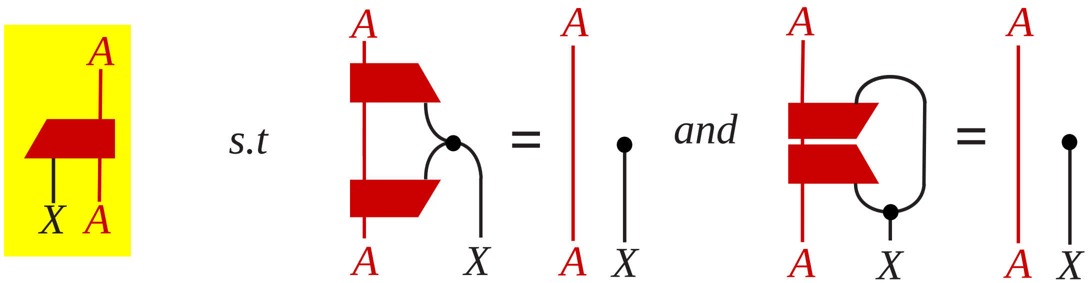
Bipartite quantum measurement:



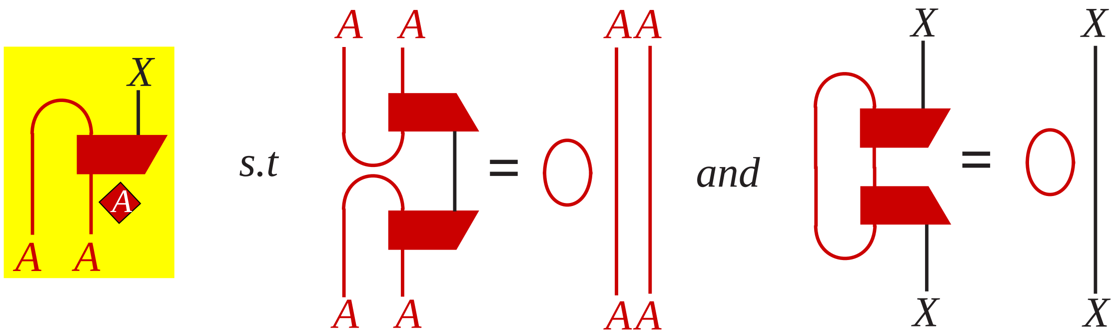


Teleportation enabling **measurement**:

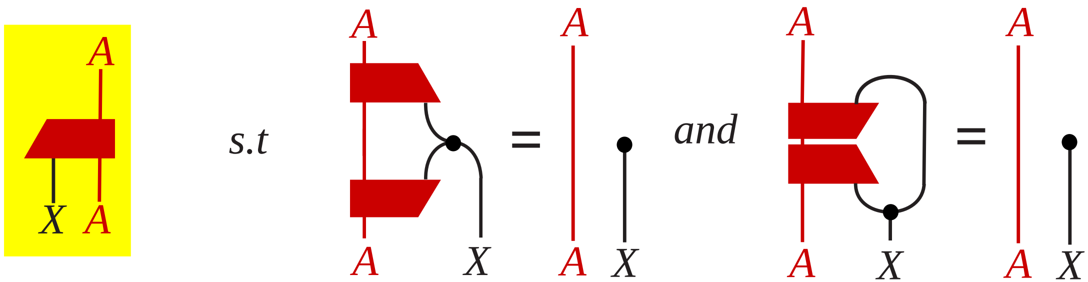




Teleportation enabling measurement:



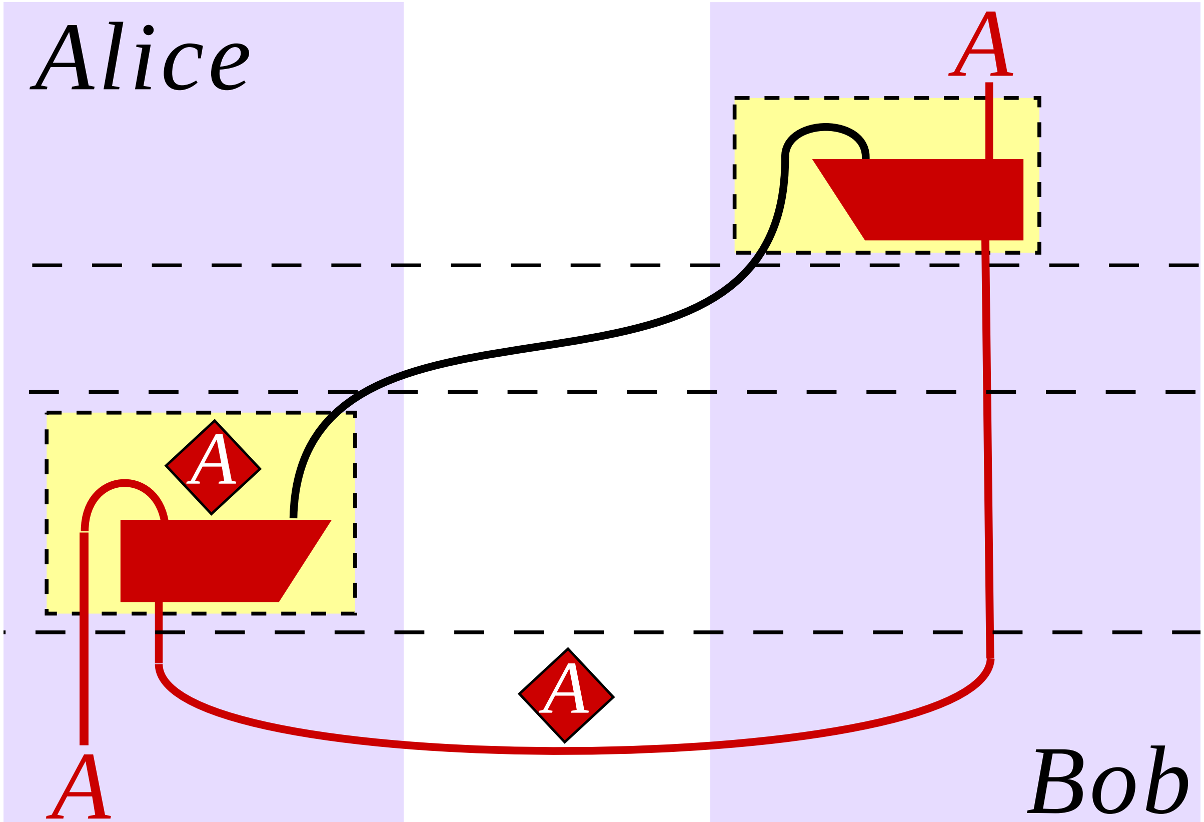
abstracts $\dim(X) \geq (\dim(A))^2$ **and** $\mathrm{Tr}(U_x \circ U_y^\dagger) = \delta_{xy}$**.**



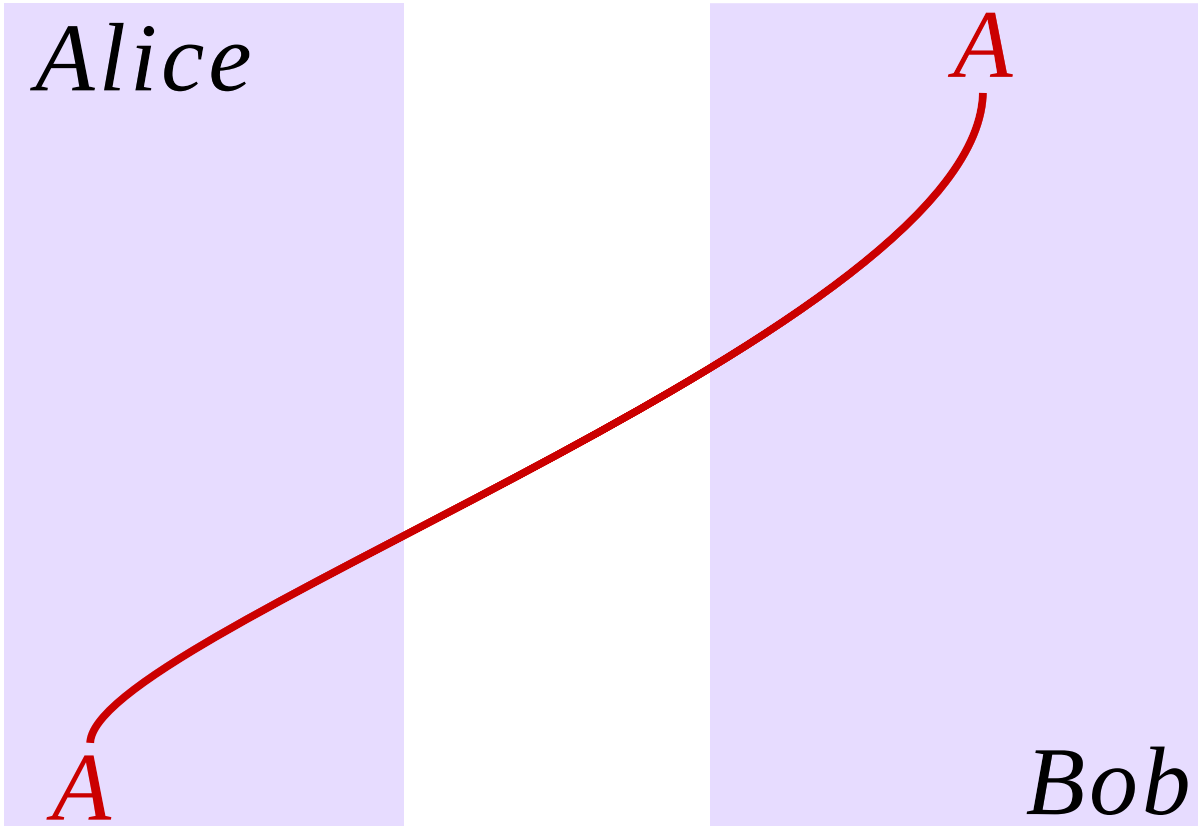
abstracts unitarity of $\{U_x\}_x$ **i.e.** $U_x^\dagger \circ U_x = U_x \circ U_x^\dagger =$

1_A.

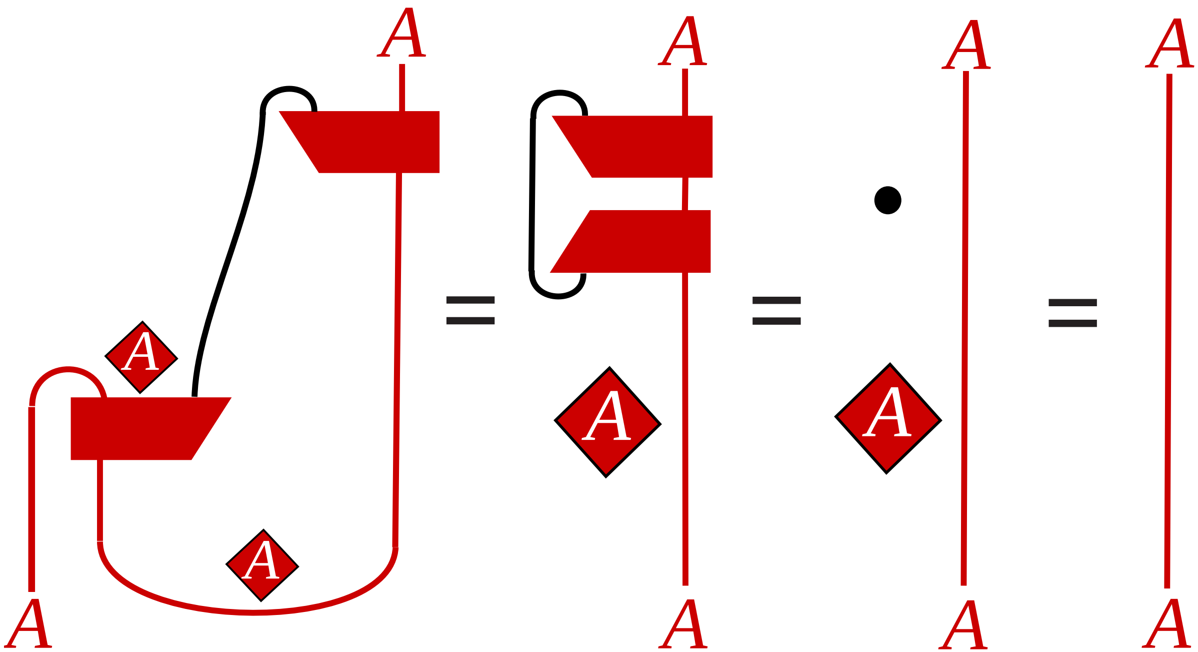
Teleportation:



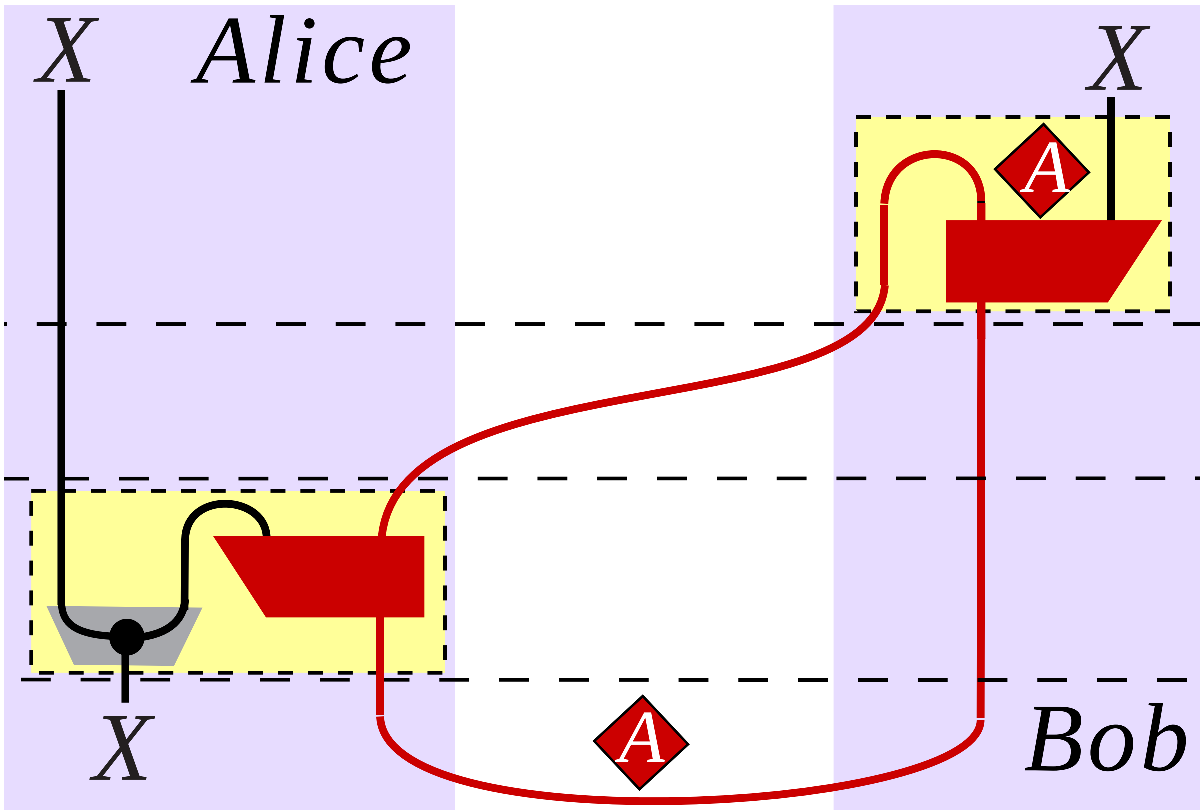
Intended behavior:



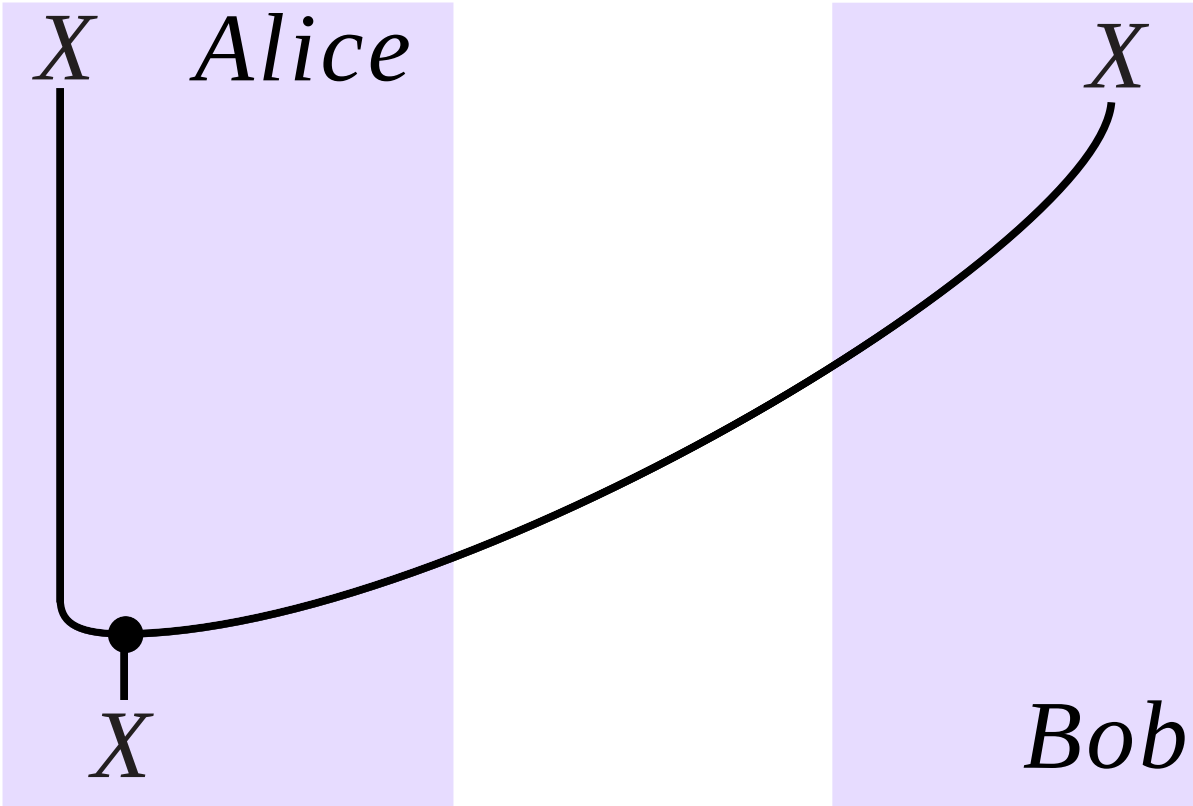
Proof:



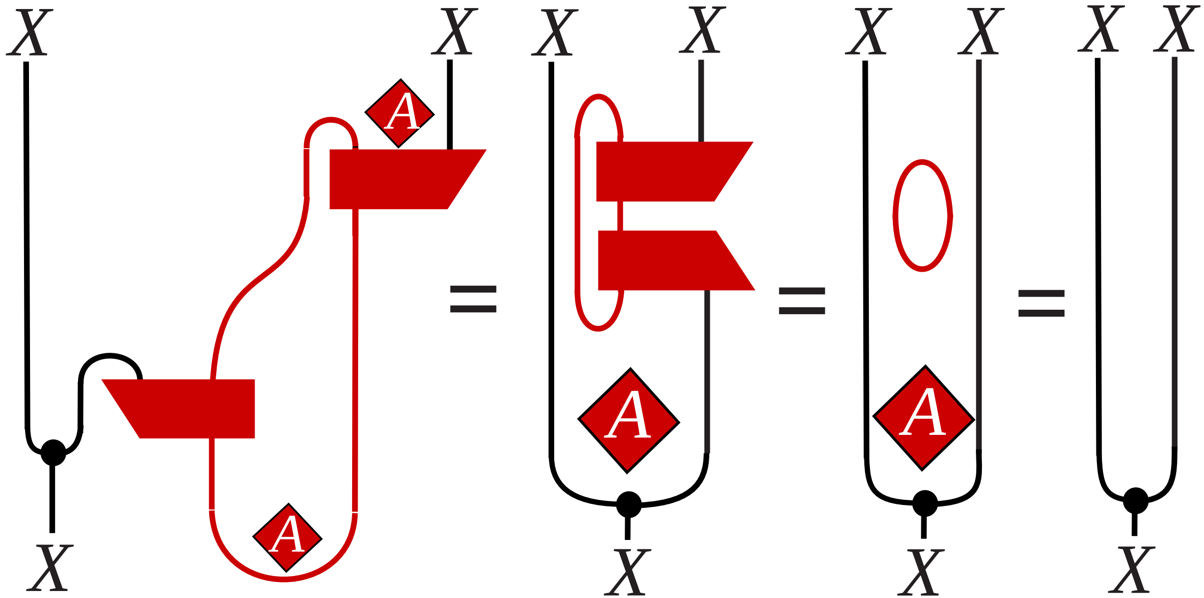
Dense coding:



Intended behavior:



Proof:



Other things we can do:

- proof correctness, generalize and find required structural resources of measurement based schemes's.
- CPMs, POVMs and Naimark's extension theorem
- resource inequalities e.g. coherent communication

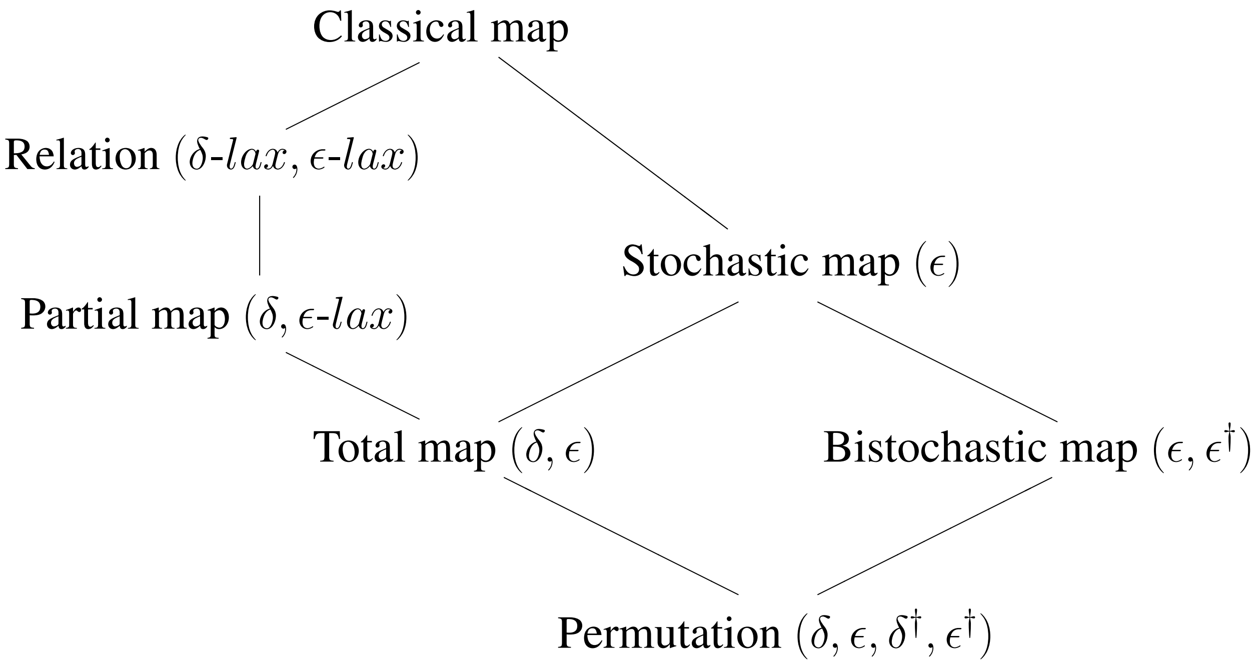
Ross Duncan's talk:[1]

- computing with basic quantum gates
- prove universality of one-way computing
- compute the quantum Fourier transform

---

[1] EXPOSES THE COMPUTATIONAL POWER OF MULTIPLE CLASSICAL CONTEXTS i.e. WE CAN USE CLASSICAL STRUCTURE NOT ONLY FOR CONTROL BUT ALSO FOR "RAW" QUANTUM CALCULUS

Classical species:



Classical maps are broadcast-able maps