Quantum Metrology

in open quantum systems

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Parameter estimation

This yields ERROR

classical estimation

unbiased estimator: $\langle E(x_1, \ldots, x_N) \rangle_\lambda = \lambda$

$$\Delta E = E(x_1, \ldots, x_N) - \langle E(x_1, \ldots, x_N) \rangle_\lambda$$

$$\partial_{x_i} \langle \Delta E(x_1, \ldots, x_N) \rangle_\lambda = 0$$

$$\langle \partial_\lambda E \rangle_\lambda = \sum_{i=1}^{N} \int_{\infty}^{\infty} \Delta E \partial_\lambda p(x_1, \ldots, x_N|\lambda) \frac{\partial_\lambda p(x_i|\lambda)}{p(x_i|\lambda)} dx_1 \ldots dx_N$$

using Cauchy-Schwarz's inequality

$$\sqrt{\frac{\langle (\Delta E)^2 \rangle}{|\langle \partial_\lambda \langle E \rangle_\lambda \rangle|^2}} \geq 1/\sqrt{NF(\lambda)}$$

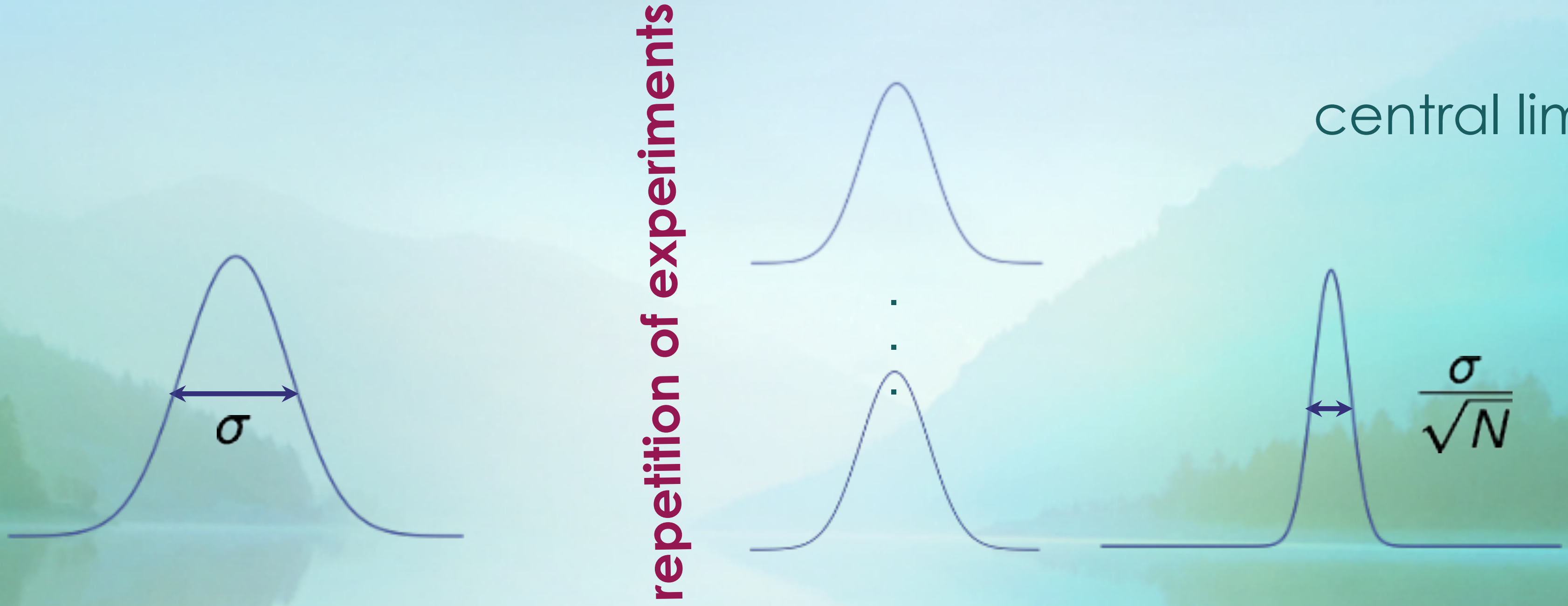
Cramer-Rao inequality

$$F(\lambda) = \int_{-\infty}^{\infty} \frac{1}{p(x|\lambda)} \left(\partial_x p(x|\lambda)\right)^2 dx$$

Fisher information

classical estimation

central limit theorem



estimator: mean value of the measurement outcomes

quantum estimation

probability of the outcome x: $p(x|\lambda) = \mathrm{Tr}[\Pi_x \varrho(\lambda)]$

quantum Fisher information:
[independent of measurements] $F^{(Q)}(\lambda, N)$

quantum Cramer-Rao inequality

$$\delta\lambda \geq 1/\sqrt{F^{(Q)}(\lambda, N)}$$

symmetric logarithmic derivative (SLD) $(1/2)[L_\lambda \varrho(\lambda) + \varrho(\lambda) L_\lambda] = \partial_\lambda \varrho(\lambda)$

for pure states $L_\lambda = 2\partial_\lambda \varrho(\lambda)$

QFI $F^{(Q)} = \mathrm{Tr}[\varrho(\lambda) L_\lambda^2]$

quantum estimation in closed systems

$$\varrho(0) \longrightarrow \boxed{e^{-i\lambda t \mathsf{H}}} \longrightarrow \varrho(\lambda)$$

$$H(\lambda) = \lambda \mathsf{H}$$

$$\partial_\lambda \varrho(\lambda) = -it[\mathsf{H}, \varrho(\lambda)]$$

$$F^{(Q)} \leq 4t^2(\Delta \mathsf{H})^2 \equiv 4t^2 \mathrm{Cov}_\varrho(\mathsf{H}, \mathsf{H})$$

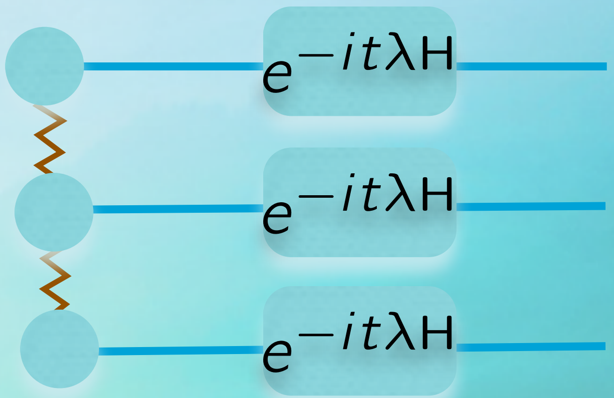
$$\boxed{\Delta\lambda \geq 1/(2t\Delta \mathsf{H})}$$

quantum estimation in closed systems

entanglement

Heisenberg limit

$$\Delta \mathrm{H} \to N \Delta \mathrm{H}$$



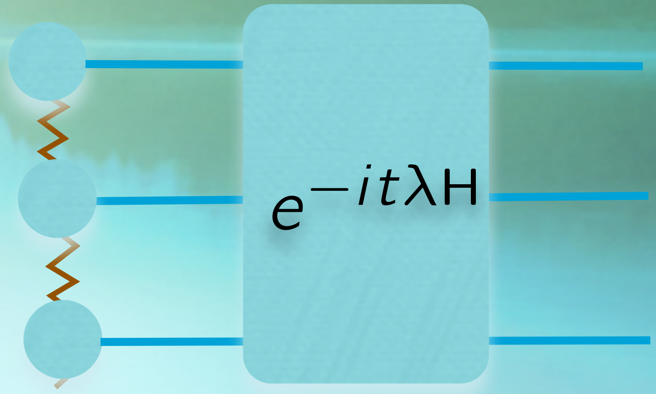
[V. Giovannetti, S. Lloyd, and L. Maccone, PRL (2006)]

k-body interaction

$$\mathrm{H} = \lambda \sum_{i_1,\ldots,i_k} h_{i_1,\ldots,i_k}$$

beyond the Heisenberg limit

$$\Delta \mathrm{H} \to N^k \Delta \mathrm{H}$$



[Boixo et al., PRL (2007)]

calculation of QFI is difficult for mixed states

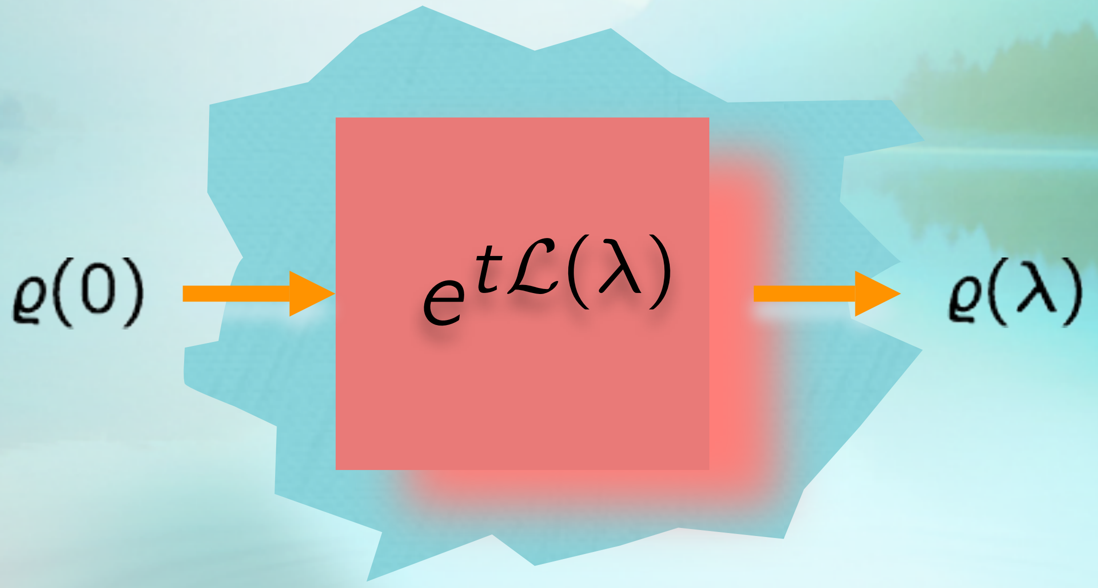
approaches:

- purification with an ancilla [Escher et al., Nature Physics (2011)]
- purification with vectorization [SA, M. Mehboudi, and A. T. Rezakhani, PRL (2014)] [F. Benatti, SA, and A.T. Rezakhani, New J. Phys. (2014)]
- extending the definition of SLD to non-Hermitian domain
- calculating QFI in terms of the QFI of compatible decompositions
- ... [SA, arXiv:1403.8033]

general dynamics: $\varrho(t,\lambda) = \sum_i A_i(t,\lambda)\varrho(0)A_i^\dagger(t,\lambda)$

Kraus representation

dynamics with a generator: $\varrho(t,\lambda) = e^{t\mathcal{L}(\lambda)}[\varrho(0)]$



purification with an ancilla

$$\mathcal{F}^{(Q)} = 4\min_{\{A_i\}}\left(\langle \mathsf{H}_1\rangle_{\varrho_0} - \langle \mathsf{H}_1\rangle^2_{\varrho_0}\right)$$

$$\mathsf{H}_1 = \sum_i \partial_\lambda A_i^\dagger \partial_\lambda A_i$$
$$\mathsf{H}_2 = i\sum_i A_i^\dagger \partial_\lambda A_i$$

- **for any set of Kraus operators one may find an UPPER bound on QFI**
- **the bound is achievable, but it needs optimization**

[Escher et al., Nature Physics (2011)]

dissipative Cramer-Rao bound (in terms of generator)

for simplification:

$$\mathcal{L}(\lambda) = \lambda \mathsf{L}$$

vectorization:

$$|\varrho(t,\lambda)\rangle\rangle = e^{t\widetilde{\mathcal{L}}(\lambda)}|\varrho(0)\rangle\rangle$$

normalized pure state

$$\widetilde{\varrho}(t;\lambda) = |\varrho(t;\lambda)\rangle\rangle\langle\langle\varrho(t;\lambda)|/\mathrm{Tr}[\varrho^2(t;\lambda)]$$

SLD:

$$\widetilde{L}_\lambda = 2\partial_\lambda \widetilde{\varrho}$$

generalized QFI:

$$\boxed{\widetilde{\mathcal{F}} = 4t^2 \mathrm{Cov}_{\widetilde{\varrho}}[\widetilde{\mathsf{L}}^\dagger, \widetilde{\mathsf{L}}]} \longrightarrow \boxed{\text{error} \propto 1/\Delta\text{generator}}$$

$$\delta\lambda \propto 1/\Delta H$$

$$\mathrm{Cov}_{\widetilde{\varrho}}[\widetilde{\mathsf{L}}^\dagger, \widetilde{\mathsf{L}}] = \mathrm{Tr}[\widetilde{\varrho}\widetilde{\mathsf{L}}^\dagger\widetilde{\mathsf{L}}] - \mathrm{Tr}[\widetilde{\varrho}\widetilde{\mathsf{L}}^\dagger]\mathrm{Tr}[\widetilde{\varrho}\widetilde{\mathsf{L}}]$$

how dose the generalized QFI relate to the QFI of the system?

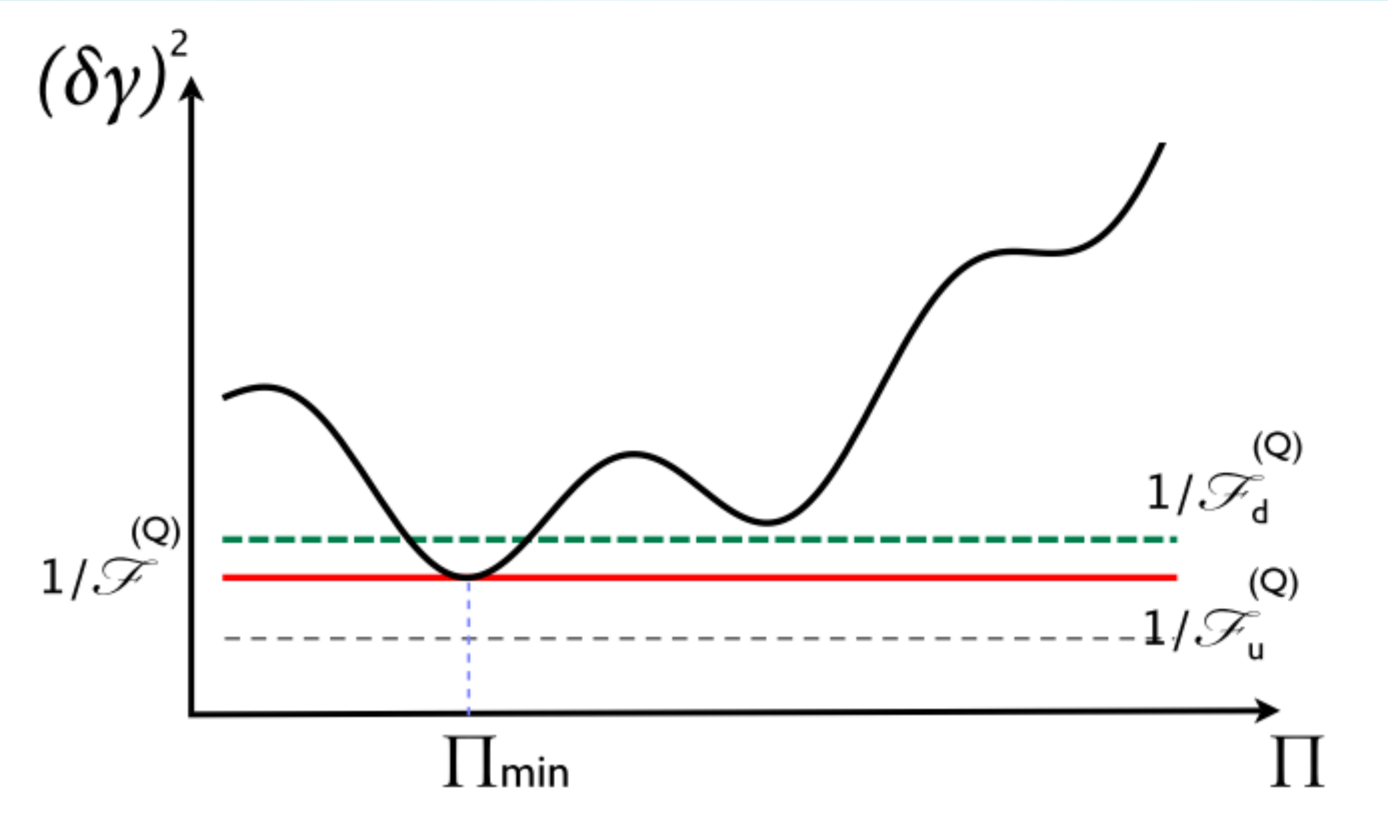
$$L_{\tilde{\varrho}} = L_{\varrho} \otimes I + I \otimes L_{\varrho}^{T} - \partial_x \ln \mathrm{Tr}[\varrho^2]$$

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})} = \frac{2}{\mathrm{Tr}[\varrho^2]}\left(\mathrm{Tr}[\varrho L_{\varrho} \varrho L_{\varrho}] + \mathrm{Tr}[\varrho^2 L_{\varrho}^2] - 2\frac{(\mathrm{Tr}[\varrho^2 L_{\varrho}])^2}{\mathrm{Tr}[\varrho^2]}\right)$$

a LOWER bound on the QFI

$$\frac{\mathrm{Tr}[\varrho^2]}{4\lambda_{\mathrm{max}}(\varrho)}\widetilde{\mathcal{F}}^{(\mathrm{Q})} \leqslant \mathcal{F}^{(\mathrm{Q})} \leqslant \frac{\mathrm{Tr}[\varrho^2]}{4\lambda_{\mathrm{min}}(\varrho)}\widetilde{\mathcal{F}}^{(\mathrm{Q})} + F(\varrho)$$

LOWER bound vs. UPPER bound



non-Hermitian SLD (dynamical bound)

motivation:

time derivative of ϱ $\quad \partial_\tau \varrho = -i[\mathrm{x}_0 H, \varrho] + \sum_i \mathrm{x}_i \Gamma_i \varrho \Gamma_i^\dagger - \frac{1}{2}\{\Gamma_i^\dagger \Gamma_i, \varrho\}$

$\equiv$ **non-Hermitian**

$$\partial_\tau \varrho = D_\varrho \varrho + \varrho D_\varrho^\dagger$$

where $D_\varrho := -i\mathrm{x}_0 H - \frac{1}{2}\sum_i \mathrm{x}_i(\Gamma_i^\dagger \Gamma_i - \Gamma_i \varrho \Gamma_i^\dagger \varrho^{-1})$

generalization: nSLD $\partial_\mathrm{x} \varrho = (L_\varrho \varrho + \varrho L_\varrho^\dagger)/2$ $\Rightarrow$ $\mathcal{F}^{(\mathrm{Q})}(\mathrm{x}) \leqslant \mathrm{Tr}[L_\varrho \varrho L_\varrho^\dagger] =: \mathcal{F}_{\mathrm{u}1}^{(\mathrm{Q})}(\mathrm{x})$

NB. non-Hermitian right logarithmic derivative operators have already been used in the literature [Fujiwara and Imai, JPA (2008)] a general Lindblad equation

$$\partial_\tau \varrho = -i[\mathrm{x}_0 H, \varrho] + \sum_i \mathrm{x}_i \left(\Gamma_i \varrho \Gamma_i^\dagger - \frac{1}{2}\{\Gamma_i^\dagger \Gamma_i, \varrho\}\right)$$

estimation of x_0

$$\partial_{\mathrm{x}_0} \varrho = -i\tau[H - \langle H \rangle_\varrho, \varrho] \Rightarrow L_0 = -2i\tau(H - \langle H \rangle_\varrho)$$

$$\mathcal{F}^{(\mathrm{Q})}_{\mathrm{u1},\mathrm{x}_0} = 4\tau^2 \Delta^2_\varrho H$$

estimation of x_i

$$\partial_{\mathrm{x}_i} \varrho = \tau\left(\Gamma_i \varrho \Gamma_i^\dagger - \frac{1}{2}\{\Gamma_i^\dagger \Gamma_i, \varrho\}\right) \Rightarrow L_i = \tau(\Gamma_i \varrho \Gamma_i^\dagger \varrho^{-1} - \Gamma_i^\dagger \Gamma_i)$$

$$\mathcal{F}^{(\mathrm{Q})}_{\mathrm{u1},\mathrm{x_i}} = \tau^2 \left(\langle (\Gamma_i^\dagger \Gamma_i)^2 \rangle_\varrho - 2\langle \Gamma_i^{\dagger 2} \Gamma_i^2 \rangle_\varrho + \mathrm{Tr}[\varrho^{-1}(\Gamma_i \varrho \Gamma_i^\dagger)^2] \right)$$

dephasing quantum channel:

$$\partial_\tau \varrho = -i\mathrm{x}_0[H, \varrho] + \mathrm{x}_1(\sigma_z \varrho \sigma_z - \varrho)$$

entangled initial state: $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$

$$\mathcal{F}^{(\mathrm{Q})}_{\mathrm{u1}} = N^2\tau^2/(e^{2N\tau\mathrm{x}} - 1)$$

$$\mathcal{F}^{(\mathrm{Q})} = N^2\tau^2/e^{2N\tau\mathrm{x}}$$

kinematic bound

convex decomposition: $\varrho = \sum_i p_i \varrho_i$

$$\partial_x (p_i \varrho_i) = (1/2)[\mathrm{L}_i p_i \varrho_i + p_i \varrho_i \mathrm{L}_i]$$

[Demkowicz-Dobrzanski et al., Nature Communications (2012)] [Fujiwara, PRA (2001)]

kinematic bound on the QFI: $\mathcal{F}^{(\mathrm{Q})}_{\mathrm{u2}} = \mathcal{F}^{(\mathrm{c})}(\{p_i\}) + \sum_i p_i \mathcal{F}^{(\mathrm{Q})}_i$

classical FI of probability distribution: $\mathcal{F}^{(\mathrm{c})}(\{p_i\}) = \sum_i (\partial_{\mathrm{x}} p_i)^2 / p_i$

QFI of each state: $\mathcal{F}^{(\mathrm{Q})}_i = \mathrm{Tr}[\varrho_i L_i^2]$

transition from classical to quantum:

$$\mathcal{F}^{(\mathrm{Q})}_{\mathrm{u2}} = c_1(\alpha, q, \tau) N + c_2(\alpha, q, \tau) N^2$$

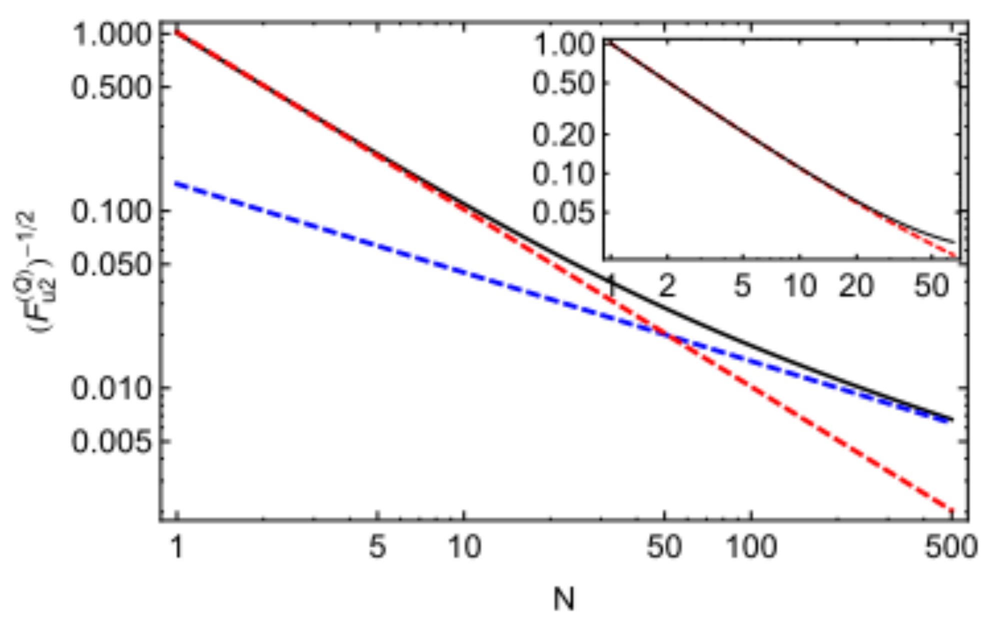
transition point: $N^* = \frac{c_1}{c_2}$

example:

$$A_1 = (\sqrt{q}\cos[\alpha\omega] + i\sqrt{1-q}\sin[\alpha\omega]\sigma_z) e^{i\omega\tau\sigma_z/2}$$

$$A_2 = (i\sqrt{q}\sin[\alpha\omega] + \sqrt{1-q}\cos[\alpha\omega]\sigma_z) e^{i\omega\tau\sigma_z/2}$$

$$\varrho(0) = |\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N|$$



kinematic upper bound + nSLD : purification with ancilla

Kraus representation $\varrho_\tau(\mathrm{x}) = \sum_i A_i(\mathrm{x},\tau)\varrho_0 A_i^\dagger(\mathrm{x},\tau)$

a natural decomposition: → $p_i = \mathrm{Tr}[A_i\varrho_0 A_i^\dagger]$ $\varrho_i = A_i\varrho_0 A_i^\dagger/\mathrm{Tr}[A_i\varrho_0 A_i^\dagger]$

using nSLD: $\partial_\mathrm{x}(p_i\varrho_i) = (\mathsf{L}_i p_i\varrho_i + p_i\varrho_i\mathsf{L}_i^\dagger)/2$

$\mathsf{L}_i A_i = 2\partial_\mathrm{x} A_i + i\alpha A_i$

applying on the kinematic bound $\mathcal{F}_{\mathrm{u2}}^{(\mathrm{Q})} = \sum_i 4(\langle\partial_\mathrm{x} A_i^\dagger\partial_\mathrm{x} A_i\rangle_{\varrho_0} - i\alpha\langle A_i^\dagger\partial_\mathrm{x} A_i\rangle_{\varrho_0}) + \alpha^2$

optimization: $\mathcal{F}_{\mathrm{u2}}^{(\mathrm{Q})}(\mathrm{x}) = 4\Big(\langle\mathsf{H}_1\rangle_{\varrho_0} - \langle\mathsf{H}_2\rangle_{\varrho_0}^2\Big)$

where: $\mathsf{H}_1 = \sum_i\partial_\mathrm{x} A_i^\dagger\partial_\mathrm{x} A_i$ $\mathsf{H}_2 = i\sum_i A_i^\dagger\partial_\mathrm{x} A_i$

Thanks