Quantum random walk in two dimensions

(Effects of translation operator on probability distribution)

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What is Quantum Random Walk?

- **Classical:** random walk in one dimension describes a particle that moves in the positive or negative direction according to the random outcome of some unbiased binary variable (e.g., a fair coin).
- **Quantum:** The particle with two degree of freedom (for example spin) can move left or right according to its spin and one unitary operator can play coin roles (for example Hadamard operator).In quantum random walk (QRW) we have 2 operator, coin operator **(C)** and translation operator **(S)**.C act on Hilbert space of particle's spin (coin space) H_C and S act on Hilbert space of particle's position H_P. C makes superposition of particle's spin and S moves particle left or right according to particle's spin. The QRW of ***n*** step defined as the transformation U^n, where U, acting on $H = H_C \otimes H_P$. $U = S.(C \otimes I)$,

If the initial state be in origin with spin down and our coin be Hadamard**(H).**

$$C = H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\ S = \exp(-2iS_z \otimes P)\ \ and\ \ |\psi_0\rangle = |0\rangle \otimes |-\rangle$$

$$1- \ H \otimes I|\psi_0\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle) \otimes |0\rangle$$

$$2- \ S.(H \otimes I) = S.\left(\frac{1}{\sqrt{2}}(|-\rangle + |+\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(|-\rangle \otimes |-1\rangle + |+\rangle \otimes |1\rangle)$$

In general, after ***n*** step the state of particle is.

$$|\psi_n\rangle = U^n|\psi_0\rangle$$

Quantum random walk can be different with classical random walk because of existence of superposition in quantum which is impossible in classic.

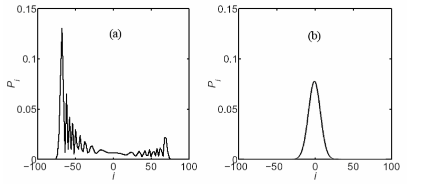

Fig.1: (a) QRW probability distribution with 100 steps and (b) classic RW with 100 steps

Difference between CRW and QRW

- CRW on the line after T step has variance $\sigma^2 = T$, so the expected distance from the origin is of order $\sigma = T^{1/2}$ but the QRW has variance that scales with $\sigma^2 \propto T^2$, which implies that the expected distance from the origin is of order $\sigma \propto T$ ***(The QRW propagates quadratically faster!)***[1].
- The QRW spreads roughly uniformly over the positions in the interval $[-T^{1/2}, T^{1/2}]$. This is again in stark contrast to CRW case in which the distribution is peaked around the origin and drops off exponentially[2].

Quantum Random Walk in two dimensions

In two dimensions QRW, one particle is free to move in two directions according to its initial spins. So we require defining generalizations of coin (C) and generalization of translation operator (S) to act on $H_2 \otimes H_2$ and two dimensional lattices H_P respectively and our initial state should has two separate particles which this particles spins define direction of movement in two directions. We can generalization these as follows.

$$C = H \otimes H = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}, \qquad S = \exp(-2iS_z^1 \otimes P_x - 2iS_z^2 \otimes P_y)$$

Operator C act on spins space and makes superposition for each spin independently and operator S according to first spin move particle along ***X*** axes and according to second spin, move particle along ***Y*** axes.

In QRW we have three items that they affect the probably distribution.

1. ***Initial state***
2. ***Coin***
3. ***Translation operator***

In two dimensional state we can take the initial state as *separable state*, *symmetry state*,… and even *entangle state* which is impossible in one dimensional QRW.

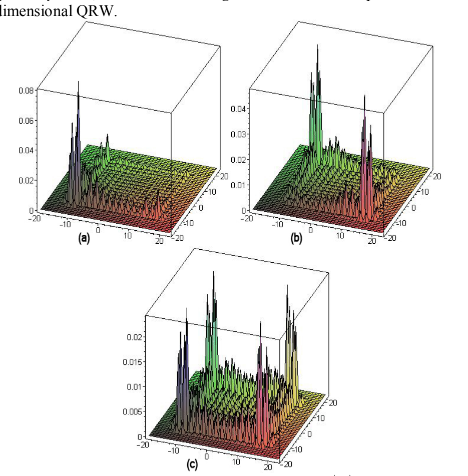

Fig.2: Probability distribution with Hadamard coin and initial state: **(a)** $|-,-\rangle$, **(b)** Entangled state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle - |+,-\rangle)$ and **(c)** Symmetry state $|\Phi_s\rangle = \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle) \otimes \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle)$

We can compare the effects of initial state on probability distribution.

On the other hand we can take another coin for QRW. The must famous coins are discrete Fourier transform (**DFT**) and Grover coin (**G**) that define as follows [3].

$$\mathrm{DFT}|\mu\rangle = \frac{1}{2}\sum_{\nu=0}^{3} e^{2\pi i\mu\nu/4}|\nu\rangle$$

$$\mathrm{G}|\mu\rangle = \frac{1}{2}\left(-2|\mu\rangle + \sum_{\nu=0}^{3}|\nu\rangle\right) \qquad , \{|\mu\rangle, \mu = 0,1,2,3\}$$

If we take $\{|\mu\rangle, \mu = 0,\ldots,3 \,|\, |0\rangle = |-,-\rangle, |1\rangle = |-,+\rangle, |2\rangle = |+,-\rangle, |3\rangle = |+,+\rangle\}$ we have

$$\mathrm{DFT} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{pmatrix}, \quad \mathrm{G} = \frac{1}{2}\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{pmatrix}$$

These two operators aren't separable and we never can write them in cross form. These operators can produce entanglement unlike the Hadamard coin.

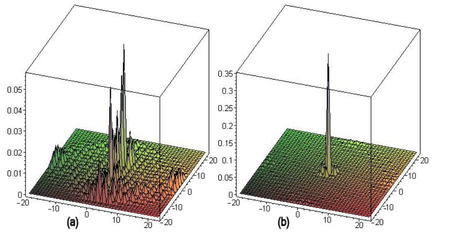

Fig.3: Probability distribution for Symmetry state $|\Phi_s\rangle = \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle) \otimes \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle)$ with: **(a) DFT** coin and **(b) G** coin

You can compare Fig.3 and (c) in Fig.2 together to understand difference of distribution with affect of different coin.

Standard deviation is a good parameter that show the expected distance from the origin and we can take the number of step as time and define the $\Delta\sigma/\Delta t$ as ***speed of spread.***

I calculate standard deviation with Hadamard,DFT and G coin for different initial state.

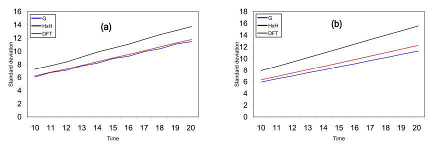

Fig.4: time dependence of Standard deviation with Hadamard,DFT and G coin for initial state **(a)** separable state $|-,-\rangle$ and **(b)** entangled state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle - |+,-\rangle)$

As we mentioned before, the standard deviation is linearly dependence with time (number of state) and you can see this in Fig.4 and Table.1 and it shows that the DFT and G decrease the Standard deviation. *"This suggests that the entanglement between the spatial degrees of freedom serves to reduce the rate of spread"*.

	H⊗H	DFT	G
$\lvert -,-\rangle$	0.66143	0.56052	0.54546
$\lvert \Psi^-\rangle$	0.76534	0.59377	0.53538
$\lvert \Phi_s\rangle$	0.75983	0.61806	0.59167

Table.1: The slope of standard deviation as function of time

We can see that both of initial state and coin affect the probability distribution and speed of spread.

We will show that the translation operator can affect the probability distributions too.

Effects of translation operator on QRW

You saw in translation operator the first spin determined the move direction along X axis and second spin did it along Y axis. These axes were independent. (interference between particle positions made distribution). Therefore we had only two one dimensional QRW in two separate axes.

If we can use first spin as condition that determine the axes, we will have QRW that axes is random and in fact we will have two dimensional QRW with RW on position and RW on axes.

We can do it with defining "***C****onditional* ***T****ranslation* ***O****perator*" (CTO) as follows.

$$S_{CTO} = e^{\left(-i(I-\sigma_z^1)\otimes S_z^2 \otimes P_y - i(I+\sigma_z^1)\otimes S_z^2 \otimes P_x\right)}$$

where $\sigma_i \quad i = x,y,z$ are pauli matrices and $S_i \quad i = x,y,z$ are spin operator. This operator move particle along X axis according the second spin when the first spin be (+) and move particle along Y axis according the second spin when the first spin be (-).

$$Conditional\begin{cases} S_{CTO}|-,-\rangle \otimes |i,j\rangle = |-,-\rangle \otimes |i,j-1\rangle \\ S_{CTO}|-,+\rangle \otimes |i,j\rangle = |-,+\rangle \otimes |i,j+1\rangle \\ S_{CTO}|+,-\rangle \otimes |i,j\rangle = |+,-\rangle \otimes |i-1,j\rangle \\ S_{CTO}|+,+\rangle \otimes |i,j\rangle = |+,+\rangle \otimes |i+1,j\rangle \end{cases} \qquad Normal\begin{cases} S|-,-\rangle \otimes |i,j\rangle = |-,-\rangle \otimes |i-1,j-1\rangle \\ S|-,+\rangle \otimes |i,j\rangle = |-,+\rangle \otimes |i-1,j+1\rangle \\ S|+,-\rangle \otimes |i,j\rangle = |+,-\rangle \otimes |i+1,j-1\rangle \\ S|+,+\rangle \otimes |i,j\rangle = |+,+\rangle \otimes |i+1,j+1\rangle \end{cases}$$

By compare normal translation operator with conditional translation operator we can see that in normal case the particle move along both of X and Y axes in each step but in conditional case only one movement along one axis happen in each step. Therefore we will expect that CTO decrease expected distance (standard deviation).

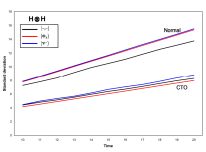

Fig.5: compare of Normal and CTO with different initial state

But we can show that the CTO can make entanglement for separable state and increase it for entangled state in compare with normal translation operator.

To show the effect of CTO on entanglement we should introduce one measure for measurement of entanglement. One of good measure for family of two qubit states is ***concurrence*** which defines as follows [4].

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Where the λ_i are the non-negative eigenvalues, in decreasing order, of Hermitian matrix $R \equiv \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ and $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$

When it is expressed in standard basis we can use concurrence $C(\rho)$ as a measure of entanglement. For pure state $|\psi\rangle = a_1|-,-\rangle + a_2|-,+\rangle + a_3|+,-\rangle + a_4|+,+\rangle$, the concurrence takes the form

$$C(\psi) = |\langle\psi|\tilde{\psi}\rangle| = 2|a_1a_4 - a_2a_3|$$

Each point of two dimensional lattices in QRW is pure state; therefore we can calculate concurrence for each point.

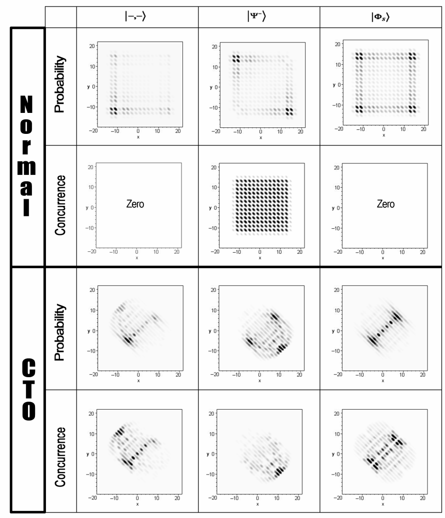

Fig.6: compare of concurrence and probability distribution for CTO and normal operator for different initial states $|-,-\rangle$, Entangled state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle - |+,-\rangle)$ and Symmetry state $|\Phi_s\rangle = \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle) \otimes \frac{1}{\sqrt{2}}(|-\rangle + i|+\rangle)$

You can see in Fig.6 CTO can produce entanglement for separable state and it is very interesting that in CTO case the more probable state has more concurrence unlike the normal case.

In normal case even you take the entangled state as initial state the more probable state has less concurrence and vice versa.

We can define ***expected concurrence*** as a follows which determine average amount of concurrence when you choice randomly many point from lattice.

$$EC(|\psi_n\rangle) = \sum_{x,y=-n}^{n} P(x,y)C(|x,y\rangle_n) \qquad |x,y\rangle_n = I \otimes \langle x,y|U^n|\psi_0\rangle$$

Where $P(x,y)$ is probability of finding particle in (x,y) and $|x,y\rangle_n$ is projection of particle states after n step in (x,y).

I calculate expected concurrence state and speed of spread ($\Delta\sigma/\Delta t$) for different initial state and summarize them in Table.2 for compare.

H⊗H		$\lvert -,-\rangle$	$\lvert \Psi^-\rangle$	$\lvert \Phi_s\rangle$
Normal	Speed of spread	0.661426	0.765344	0.759828
	Expected concurrence	0	0.000543	0
CTO	Speed of spread	0.394747	0.430234	0.385983
	Expected concurrence	0.005615	0.005835	0.002412

Table.2: speed of spread and concurrence for different initial state in Normal and CTO case.

Conclusions

As mentioned before quantum random walk has three parts that affect probability distribution and entanglement. The conditional translation operator (CTO) can change the probability distribution and entanglement (concurrence).

We saw that CTO decrease the expected distance from origin $\Delta\sigma$ and speed of spread ($\Delta\sigma/\Delta t$) it is result of one movement in one axis in each step in CTO unlike two movements in two axes in Normal case.

All $\Delta\sigma$ is linear in time same as Normal case but its slope is less than Normal case and proportional to slope of Normal case but the proportional constant depends on initial state (Fig.5).

The interesting feature of CTO is creating entanglement for separable state and its probability distribution is so that probable state (point) has more entanglement (Fig.6).

References

[1] J.Kempe, *Quantum random walks-an introductory overview*, quant-ph/0303081.
[2] A.Ambainis,E.Bach,A.Nayak,A.Vishwanath,J. Watrous,One-dimensional quantum walks. Proceedings of STOC'01.
[3] T.D.Mackay, S.D.Bartlett, L.T.Stephenson,and B.C.Sanders, *Quantum walks in higher dimensions*, J.Phys.A : Math. Gen. 35 (2002) 2745-2753.
[4] W. K. Wootters, Phys. Rev. Lett .80, 2245 (1998).