Entanglement and squeezing of a four qubit system via two-axis countertwisting Hamiltonian

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Abstract

We study entanglement and squeezing properties of a spin system of 4 qubits under the influence of the two-axis counter twisting Hamiltonian. Our initial spin state is a coherent one in the z-direction, which evolves in time. We choose the squeezing parameters given by Wineland and Kitagawa as the criteria of spin squeezing. The criterion of pairwise entanglement is chosen to be the concurrence and that of the bipartite entanglement, the linear entropy. We will study and plot the time dependence of the squeezing and entanglement parameters and also determine the time domains in which squeezing and entanglement properties can or can not exist simultaneously.

1.Introduction

Spin squeezing [1-15] has applications in several fields of physics; among them we mention of interferometers and precision spectroscopy [16-19]. It is also closely related to quantum entanglement; therefore, it is relevant in quantum information and computation too [20-29]. In this work, we consider a spin system consisting of 4 qubits (four one-half spins), which is initially in a coherent state and study its time evolution via the well known two-axis countertwisting Hamiltonian [30]

$$H = \frac{\chi}{2i}\left(\hat{S}_{+}^{\,2} - \hat{S}_{-}^{\,2}\right). \tag{1}$$

Obtaining the time dependent spin operators, we investigate the squeezing properties of the system, using the spin squeezing parameter

$$\xi_K^2 = \frac{2(\Delta S_{n_\perp})^2_{\min}}{S}, \tag{2}$$

introduced by Kitagawa et al and also the squeezing parameter

$$\xi_W^2 = 2S(\Delta S_{n_\perp})^2_{\min} \Big/ \left|\left\langle \vec{\hat{S}} \right\rangle\right|^2, \tag{3}$$

introduced by Wineland et al [31]. Here $\hat{n}_\perp$ represents a direction perpendicular to the mean spin direction $\hat{n} = \left\langle \vec{\hat{S}} \right\rangle \Big/ \sqrt{\left\langle \vec{\hat{S}} \right\rangle \cdot \left\langle \vec{\hat{S}} \right\rangle}$. We also study the entanglement properties of this system, considering linear entropy

$$E_L = 1 - Tr(\rho_i^2), \tag{4}$$

as a measure of bipartite entanglement (the entanglement between one spin and all the others)[23], where ρ_i is the reduced density matrix for the ith particle; and the concurrence

$$C = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{5}$$

as the measure of pairwise entanglement (the entanglement of a pair of spins)[32,33]. Here, λ_i are the eigenvalues of the 4 by 4 matrix

$$\mathrm{P}_{ij} = \rho_{ij}(\sigma_{iy} \otimes \sigma_{jy})\rho_{ij}^{*}(\sigma_{iy} \otimes \sigma_{jy}), \tag{6}$$

where, ρ_{ij} is the reduced density matrix element and ρ_{ij}^{*} is its complex conjugate.

2. Spin squeezing

A general spin S coherent state, is given by

$$\left|\theta,\phi\right\rangle = \left(1+\tan^2\frac{\theta}{2}\right)^{-S} \times \sum_{m=0}^{2S}\left(e^{i\phi}\tan\frac{\theta}{2}\right)^{m}\binom{2S}{m}^{1/2}\left|S,S-m\right\rangle_z , \tag{7}$$

where, θ and ϕ are the polar and the azimuthal angle, respectively and $\left|S,S-m\right\rangle_z$ are eigenstates of S^2 and S_z [34]. This may be considered as an ensemble of $N=2S$ qubits (N one-half spins) with no interaction between them for the moment. Thus, the collective spin operators in the direction $\hat{n}$, may be given by

$$S_n = \sum_{i=1}^{N}\frac{1}{2}\sigma_{i,n} . \tag{8}$$

Where, $\sigma_{i,n} = \hat{\sigma}_i.\hat{n}$ is the Pauli matrix in the $\hat{n}$ direction for the i th spin. We consider a spin 2 coherent state along the z-direction, in our investigation, as follows

$$\left|\theta=0,\phi\right\rangle = \left|S,S\right\rangle = \left|2,2\right\rangle . \tag{9}$$

We note that

$$\left\langle 2,2\right|\hat{S}_x\left|2,2\right\rangle = \left\langle 2,2\right|\hat{S}_y\left|2,2\right\rangle = 0 , \tag{10}$$

$$\left\langle \vec{\hat{S}}\right\rangle = \left\langle \hat{S}_z\right\rangle = \left\langle 2,2\right|\hat{S}_z\left|2,2\right\rangle = 2 , \tag{11}$$

$$(\Delta S_x)^2 = (\Delta S_y)^2 = 1 ; \tag{12}$$

meaning that the average spin rests along the z direction and all the spins are upward at $t=0$. Moreover the uncertainty relation

$$\left(\Delta S_x\right)^2\left(\Delta S_y\right)^2 \geq \frac{1}{4}\left|\left\langle \hat{S}_z\right\rangle\right|^2 , \tag{13}$$

with the equality sign is satisfied here.

We now study the time evolution of this system via Hamiltonian (1). The nonzero matrix elements of H are given by

$$(H)_{3,1} = (H)_{5,3} = -(H)_{1,3} = -(H)_{3,5} = i\sqrt{6}\chi ,$$
$$(H)_{4,2} = -(H)_{2,4} = 3i\chi . \tag{14}$$

Thus, our time dependent bra state is found to be

$$\left\langle\psi(T)\right| = \left\langle 2,2\right|e^{iHt} \doteq [(1/2)(1+\cos(2\sqrt{3}T)),0,(\sqrt{1/2})\sin(2\sqrt{3}T),0,(1/2)(1-\cos(2\sqrt{3}T))] , \tag{15}$$

where, the scaled time, $T = \chi t$ has been defined. Using the above dynamically generated state we find

$$\langle \psi(T)|\hat{S}_x|\psi(T))\rangle = \langle \psi(T)|\hat{S}_y|\psi(T)\rangle = 0\,, \tag{16}$$

implying that the dynamic evolution has not changed the average direction of spin and it steel stands along the z-axis. We therefore have

$$\left|\langle \psi(T)|\vec{\hat{S}}|\psi(T)\rangle\right| = \left|\langle \psi(T)|\hat{S}_z|\psi(T)\rangle\right| = 2\left|\cos(2\sqrt{3}T)\right|. \tag{17}$$

It is now worthwhile to look at the uncertainty relation at time T, we find

$$(\Delta S_x)^2 = \frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) + \sqrt{3}\sin(2\sqrt{3}T)\,, \tag{18}$$

$$(\Delta S_y)^2 = \frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) - \sqrt{3}\sin(2\sqrt{3}T)\,, \tag{19}$$

$$(\Delta S_x)^2(\Delta S_y)^2 = [7 + \cos(8\sqrt{3}T)]/8\,, \tag{20}$$

and

$$\frac{1}{4}\left|\langle \hat{S}_z \rangle\right|^2 = [\cos(2\sqrt{3}T)]^2\,. \tag{21}$$

Comparing (20) and (21) with (12) we observe a redistribution of uncertainties in different directions in this situation. For example, assuming $T = \frac{\pi}{6}$, we find

$$(\Delta S_x)^2 = 3.62\,,\ \ (\Delta S_y)^2 = 0.26\,,\ \ (\Delta S_x)^2(\Delta S_y)^2 = 0.95\,,\ \ \frac{1}{4}\left|\langle \hat{S}_z \rangle\right|^2 = 0.06\,; \tag{22}$$

Implying that uncertainty along the x-axis has increased above the quantum limit, while it has decreased below that limit along the y-axis. The inequality sign in (13) is also satisfied. In fact, we are dealing with a squeezed state at this time, and those are exactly the characteristics that we expect for such a state.

To find the best squeezing direction, we rotate the coordinate system in the $x-y$ plane by angle δ, but keep the z-axis fixed. Obviously, the uncertainties along the new coordinates are functions of δ. Let's define the direction $\hat{n}_\perp = (\cos\delta, \sin\delta, 0)$ in the $x-y$ plain. We may write

$$S_{n_\perp} = S_x \cos\delta + S_y \sin\delta\,. \tag{23}$$

Therefore we find

$$(\Delta S_{n_\perp})^2 = \frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) + \sqrt{3}\sin(2\sqrt{3}T)\cos(2\delta) \cdot \tag{24}$$

Minimizing (24) with respect to δ, we find

$$(\Delta S_{n_\perp})^2_{\min} = \frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) - \left|\sqrt{3}\sin(2\sqrt{3}T)\right|. \tag{25}$$

This result shows that, the minimum uncertainty achievable along a direction in the $x-y$ plain is a periodic function of time. This implies that although the mean spin direction remains along the z-direction, but the distribution of spin directions changes in time. To show this point more vividly, we have illustrated the quasi-probability distributions $Q = \left|\langle\theta,\varphi|2,2\rangle\right|^2$ and $Q = \left|\langle\theta,\varphi|\psi(T)\rangle\right|^2$ along with their contour plots in figures 1 and 2. The elliptic contours in figure 1, in contrast to the circular ones in figure 2, represent the redistribution of probabilities and uncertainties clearly.

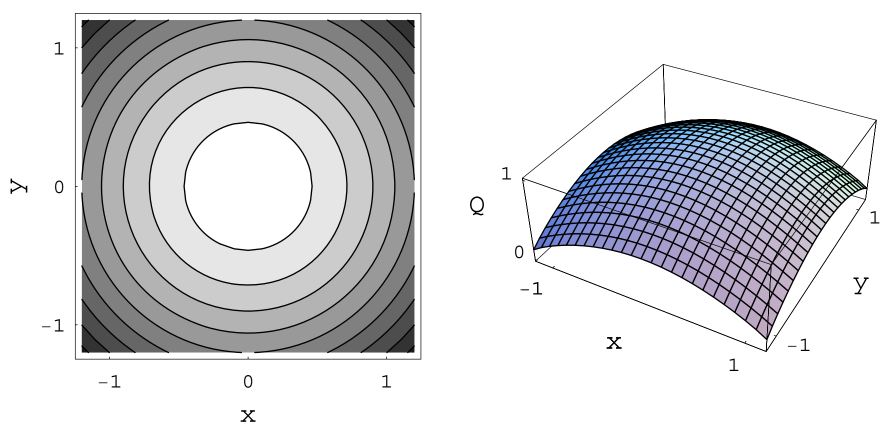

Figure 1: Quasi-probability distribution and its contour plot at $T = 0$

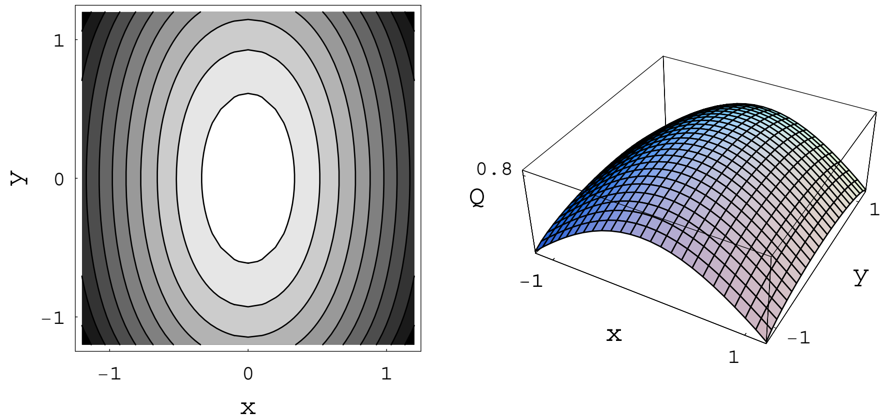

Figure 2: Quasi-probability distribution and its contour plot at $T = 0.1776$

Now, using (2), (3) and (25), we express the squeezing parameters for the $\hat{n}_{\perp}$ direction as follows.

$$\xi_K^2 = \frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) - \left|\sqrt{3}\sin(2\sqrt{3}T)\right| \ , \tag{26}$$

$$\xi_W^2 = \frac{\frac{3}{2} - \frac{1}{2}\cos(4\sqrt{3}T) - \left|\sqrt{3}\sin(2\sqrt{3}T)\right|}{(\cos(2\sqrt{3}T))^2} \ . \tag{27}$$

We have plotted these parameters as a function of T in figures 3. We note that the system becomes squeezed, according to both criteria, at the alternate time intervals, due to the dynamics provided by the Hamiltonian (1). We note that $\xi_W^2 \geq \xi_K^2$; therefore, for the values of $\xi_W^2 < 1$, that the system is squeezed according to Wineland's criterion, it is squeezed according to Kitagawa's also. The reverse is not of course always true.

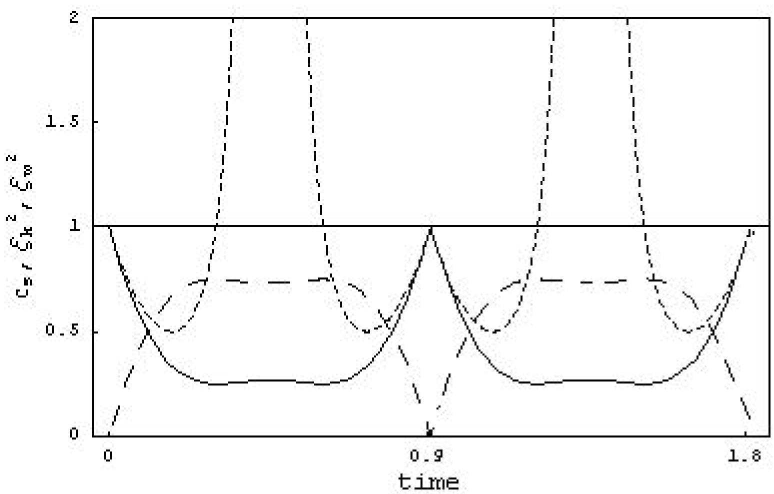

Figure 3: plots of ξ_W^2 (dotted line), ξ_K^2 (solid line) and $C_S = 3C$ (dashed line) versus time for $S = 2$
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3. Spin entanglement

First we consider the bipartite entanglement of the system. We are dealing with identical entities, therefore due to exchange symmetry the reduced matrix ρ_i^2 is the same for all the entities. Moreover, the reduced density matrix is just the one qubit density matrix and we have

$$\rho_i = \begin{pmatrix} (1+\langle\sigma_z\rangle)/2 & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & (1-\langle\sigma_z\rangle)/2 \end{pmatrix}, \tag{28}$$

where at the scaled time T we have

$$\langle\sigma_z\rangle = \cos(2\sqrt{3}T)\,,\ \langle\sigma_y\rangle = 0\,,\ \ \langle\sigma_x\rangle = 0\,, \tag{29}$$

Using (4), (28) and (29) we finally obtain

$$E_L = 1/2 - \langle\sigma_z\rangle^2/2 = (\sin(2\sqrt{3}T))^2/2\,. \tag{30}$$

Now, defining the scaled entropy $E_S = 2E_L$ [35] and eliminating time between equations (30), (26) and (27) we find

$$E_S = 2E_L = (1-\eta^2) = (\sin(2\sqrt{3}T))^2\,. \tag{31}$$

Where, η^2 which may be called squeezing ratio, has been defined by

$$\eta^2 = \frac{\xi_K^2}{\xi_W^2} \le 1\,. \tag{32}$$

This is an interesting result; The scaled entropy which is the criterion of bipartite entanglement is a linear function of the squeezing ratio η^2 and vice versa. The squeezing ratio satisfies the inequalities $0 \le \eta^2 \le 1$; thus, the scaled entropy changes in the range $1 \ge E_S \ge 0$. Moreover, smaller values of η^2 correspond to larger bipartite entanglement. We have plotted scaled entropy as a function of time in figure (4). We like to emphasize that E_S is not a simple or monotonic function of either squeezing parameters ξ_K^2 and ξ_W^2; thus, we can not relate squeezing and bipartite phenomena in a simple manner and that was the reason for introducing the squeezing ratio, in the first place.

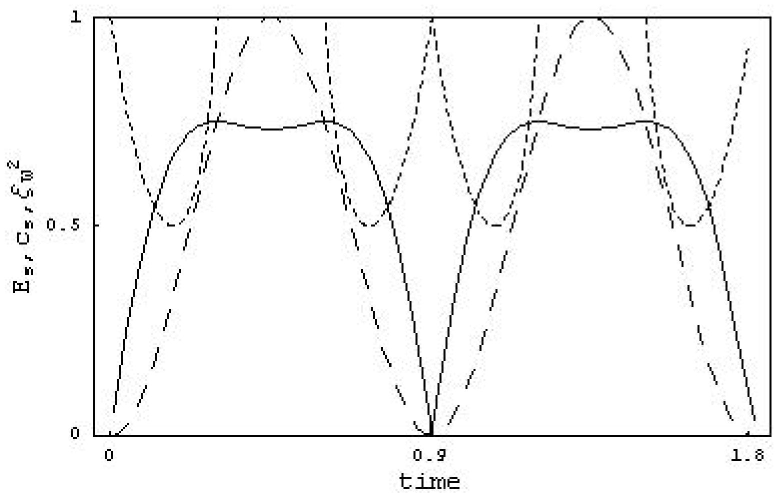

Figure 4: Plots of E_S (dashed line), C_S (solid line) and $\xi_W{}^2$ (dotted line) versus time.

We now embark upon studying the pairwise entanglement of the system. First we calculate the reduced initial density matrix ρ_{ij} at $t=0$. In fact, due to the exchange symmetry it is independent of i and j; thus we drop the indices and call it ρ for simplicity. It has only the nonzero element $\rho_{11}=1$. The nonzero matrix elements of the dynamically generated time dependent density matrix are given by[36]

$$\rho_{11} = (\cos(\sqrt{3}T))^2(2+\cos(2\sqrt{3}T))/3,$$

$$\rho_{44} = (\sin(\sqrt{3}T))^2(2-\cos(2\sqrt{3}T))/3,$$

$$\rho_{14} = \rho_{41} = (\sin(2\sqrt{3}T))/2\sqrt{3},$$

$$\rho_{22} = \rho_{23} = \rho_{32} = \rho_{33} = (\sin(2\sqrt{3}T))^2 / 6 . \qquad (33)$$

We also calculate the nonzero matrix elements of the operator $(\sigma_{iy} \otimes \sigma_{jy})$; we find

$$(\sigma_y \otimes \sigma_y)_{23} = (\sigma_y \otimes \sigma_y)_{32} = -(\sigma_y \otimes \sigma_y)_{14} = -(\sigma_y \otimes \sigma_y)_{41} = 1 . \qquad (34)$$

Using (33) and (34) in (6), we finally find the nonzero matrix elements of the operator P_{ij} , which we simply call P , as follows

$$P_{11} = [13 - \cos(4\sqrt{3}T)][\sin(2\sqrt{3}T)]^2 / 72 ,$$

$$P_{14} = 2[\cos(\sqrt{3}T)]^3[\sin(\sqrt{3}T)][2 + \cos(2\sqrt{3}T)] / 3\sqrt{3} ,$$

$$P_{41} = 2[\cos(\sqrt{3}T)][\sin(\sqrt{3}T)]^3[2 - \cos(2\sqrt{3}T)] / 3\sqrt{3} ,$$

$$P_{22} = P_{23} = P_{32} = P_{33} = [\sin(2\sqrt{3}T)]^4 / 18 . \qquad (35)$$

The square root of the eigenvalues of this matrix in the descending order are found to be

$$\sqrt{\lambda_1} = \left|(\sin(2\sqrt{3}T)) / 2\sqrt{3}\right| + \left|\sin(2\sqrt{3}T)\right|\sqrt{4 - (\cos(2\sqrt{3}T))^2} / 6 ,$$

$$\sqrt{\lambda_2} = \left|\sin(2\sqrt{3}T)\right|\sqrt{4 - (\cos(2\sqrt{3}T))^2} / 6 - \left|(\sin(2\sqrt{3}T)) / 2\sqrt{3}\right| ,$$

$$\sqrt{\lambda_3} = (\sin(2\sqrt{3}T))^2 / 3 ,$$

$$\sqrt{\lambda_4} = 0 . \qquad (36)$$

Finally, application of (36) in (5), gives us the scaled concurrence $C_S = 3C$ as a function of the scaled time T as follows

$$C_S = \sqrt{3}\left|\sin(2\sqrt{3}T)\right| - (\sin(2\sqrt{3}T))^2 . \qquad (37)$$

Eliminating time between (37) and (26) we find the following linear relation between C_S and ξ_K^2

$$C_S = 3C = [1 - \xi_K^2] . \qquad (38)$$

We have plotted the function C_S in figure (3). We observe that, if $\xi_K^2 = 1$ we have $C_S = 0$; that is if the system is not squeezed it is not pairwise entangled either. However, if $\xi_K^2 < 1$, the system is squeezed according to Kitagawa's criterion, then we have $C_S > 0$ and the system is also pairwise entangled simultaneously and vice versa.

We may also write

$$\xi_W{}^2 = \frac{1 - C_S}{1 - E_S}. \tag{39}$$

We have plotted the three functions C_S, E_S and $\xi_W{}^2$ in figure 4. For $\xi_W{}^2 = 1$ the system is not squeezed and $C_S = E_S = 0$, that is we do not have entanglement either. But, it is squeezed for $\xi_W{}^2 < 1$, which requires $C_S > E_S$; this may be considered as a criterion of the existence of squeezing in this system and vice versa.

We now eliminate time between (31) and (37) to obtain the following relation between C_S and E_S

$$C_S = \sqrt{3E_S} - E_S\ . \tag{40}$$

We have plotted C_S as a function of E_S in figure 5. Barring one maximum point at $E_S = 0.750$, it is a monotonic function of E_S; increases for the range $E_S = (0,0.75)$, while decreases for the range $E_S = (0.75,1)$.

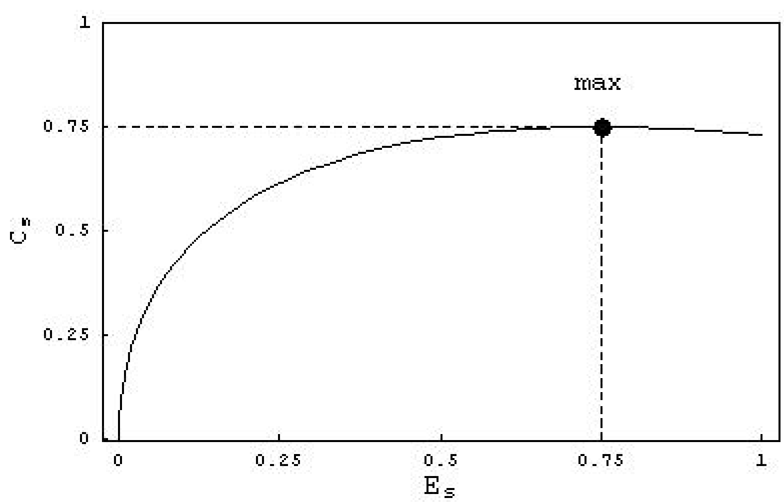

Figure 5: C_S versus E_S

4. Discussion and conclusions

We considered a four-qubit initial coherent state and studied its time evolution via the two-axis countertwisting Hamiltonian. It was observed that the average spin direction remains along the initial one, but the quasi-probability distribution for spin direction becomes asymmetrical about the z-axis, in contrast to the initial coherent case. We showed that the parameters for K-squeezing (defined by Kitagawa et al) and W-squeezing (defined by Wineland et al) are periodic functions of time. Barring some separate instances of time, the system was found to be always K-squeezed, but only W-squeezed in alternate time intervals. It was also noted that if the system is W-squeezed it will be K-squeezed also, but the reverse is not necessarily true.

We proved that the scaled entropy, which is the criterion of bipartite entanglement is a linear function of the squeezing ratio η^2 and vice versa. η^2 satisfies the inequalities $0 \le \eta^2 \le 1$; thus, the scaled entropy changes in the range $1 \ge E_S \ge 0$. Moreover, smaller values of η^2 correspond to deeper bipartite entanglement.

We also showed that pairwise entanglement is a linear function of K-squeezing parameter and the system is K-squeezed if it is pairwise entangled and vice versa.

If $\xi_W{}^2 < 1$ (W-squeezing), the inequality $C_S > E_S$ is satisfied and vice versa; thus, the latter inequality may be considered as the criterion of the existence of W-squeezing and vice versa. Finally, we showed that barring one maximum point at $E_S = 0.750$, C_S is a monotonic increasing or decreasing function of E_S.

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