Quantum-information distributors: Quantum network for symmetric and asymmetric cloning in arbitrary dimension and continuous limit

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We show that for any Hilbert-space dimension, the optimal $1 \rightarrow 2$ universal quantum cloner can be constructed from essentially the same quantum circuit, i.e., we find a universal design for universal cloners. In the case of infinite dimensions (which includes continuous variable quantum systems) the universal cloner reduces to an essentially classical device. More generally, we construct a universal quantum circuit for distributing qudits in any dimension which acts covariantly under generalized displacements and momentum kicks. The behavior of this covariant distributor is controlled by its initial state. We show that suitable choices for this initial state yield both universal cloners and optimized cloners for limited alphabets of states related by generalized phase-space displacements.

I. INTRODUCTION

One of the main tasks in quantum-information processing and quantum computing is the *distribution* of quantum information encoded in the states of quantum systems. Assume a quantum system labeled as 1 is prepared in an *unknown* pure state described by a state vector $|\Psi\rangle_1$ in an N-dimensional Hilbert space. The task is to transfer *partially* the information encoded in system 1 into a second system in a covariant way. That is, the fidelity of the operation should not depend on the particular choice of the input state $|\Psi\rangle_1$. In addition, we want to control the amount of information transferred from system 1 to system 2. One of the simplest examples of such a transformation is state swapping, when the state of system 1 is exchanged with the (known) state of system 2. In this case the complete information is transferred. Another option is to leave the system 1 in the original state. These two operations can be performed with unit fidelity irrespective of the input state of system 1. We can also consider a case intermediate between these two limiting cases, i.e., between no transfer and the complete transfer of information. One interesting version of this intermediate transformation involves the copying (cloning) of quantum information from system 1 to system 2, where, after the transformation, each of the systems 1 and 2 has the same reduced state, which is itself as close as possible to the original state $|\Psi\rangle_1\langle\Psi|$. In this case we often require that the fidelity of the information transfer does not depend on the initial state.

It is now well known that quantum information cannot be exactly copied [1]. This *no-cloning* theorem has important consequences for the whole field of quantum-information processing [2]. Nonetheless, considerable progress has been made in the development of approximate quantum cloners. Most of the effort has focused on two types of these, universal cloners that copy all input states equally well [3], and probabilistic cloners that copy a known set of states perfectly, but do so with a probability less than 1 [4]. In what follows we shall concentrate on *universal* devices.

Quantum cloning both illuminates the limits imposed by quantum mechanics on the manipulation of quantum information and can be useful in applications. It has been shown to be useful in improving the performance of imperfect quantum detectors [5] and in improving the performance of certain quantum computations [6]. In addition, it has been shown in Ref. [7] that quantum cloners can be used as optimal eavesdropping devices on the six-state cryptographic protocol. We should also note that recently an interesting cloning experiment has been proposed [8]. Moreover, two experiments have been independently reported [9,10] in which cloning of optical fields has been realized.

Universal cloners can be either symmetric or asymmetric. In a symmetric cloner the quantum information is divided equally and the output clones are identical. In an asymmetric cloner one of the clones receives more of the input quantum information than the other. Symmetric cloners were first developed to copy qubits [3,11], but have been extended to copy states in spaces of arbitrary dimension [12,13,15], and it has been proven that these cloners are optimal [16,11,17]. The study of asymmetric cloners also began with the consideration of qubits [18–20] and has been recently extended to systems of arbitrary dimension [13]. In addition, $N \rightarrow M$ cloning of coherent states (i.e., cloning in the continuous limit) has been considered [14].

What we shall do here is to exhibit a quantum circuit for symmetric and asymmetric cloners in arbitrary numbers of dimensions. In order to emphasize that what these devices do is distribute quantum-information, we shall refer to them as quantum information distributors (QID's). The circuit consists of four controlled-NOT (C-NOT) gates, or rather their generalization to N dimensions, and its form is the same for any number of dimensions. There are two inputs to this cir-

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cuit. The first is the state that supplies the information to be distributed between the two outputs. The second acts as a program and determines how the information is distributed. The infinite-dimensional version of this circuit allows us to describe quantum-information distributors for continuous variables.

Let us formulate our problem more exactly. Assume the original quantum system is in a pure state

$$|\Psi\rangle_1 = \sum_{n=0}^{N-1} c_n |x_n\rangle. \tag{1.1}$$

At the output of the quantum-information distributor we would like to have two quantum systems each with a state described in a *covariant* form,

$$\hat{\rho}_1^{(\mathrm{out})} = (1 - \beta^2)|\Psi\rangle_1\langle\Psi| + \frac{\beta^2}{N}\hat{1}_1,$$

$$\hat{\rho}_2^{(\mathrm{out})} = (1 - \alpha^2)|\Psi\rangle_2\langle\Psi| + \frac{\alpha^2}{N}\hat{1}_2, \tag{1.2}$$

where the real parameters α and β quantify the amount of information transferred from one system to the other. In particular, if $\beta=0$, then *no* information has been transfered from the original system, while, if $\alpha=0$, then all of the information in system 1 has been transferred to system 2. The parameters α and β are related (see below). From the covariant form of the output density operators it follows that the fidelity of the information transfer is input-state independent. The terms proportional to $\hat{1}/N$ in the density operators describe the amount of noise introduced into the systems at the output by the information transfer process.

Our task in this paper is to develop a quantum circuit for the universal quantum-information distributor for arbitrary-dimensional quantum systems. In Sec. II we start our discussion with the mathematical formalism needed to investigate our problem. Then in Sec. III we present a quantum network for the universal quantum-information distributor, while in Sec. IV we generalize the discussion to continuous variables. Finally, in Sec. V we summarize our results.

II. FROM DISCRETE TO CONTINUOUS VARIABLES

In order to make the discussion self-contained we first present a brief review of the formalism describing quantum states in a finite-dimensional Hilbert space. Here we follow the notation introduced in Refs. [21,22] (see also Ref. [23]). Let the N-dimensional Hilbert space be spanned by N orthogonal normalized vectors $|x_k\rangle$ and equivalently by N vectors $|p_l\rangle$, $k,l=0,\ldots,N-1$, where these bases are related by the discrete Fourier transform

$$|x_k\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \exp\left(-i\frac{2\pi}{N}kl\right)|p_l\rangle,$$

$$|p_l\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(i\frac{2\pi}{N}kl\right)|x_k\rangle. \tag{2.1}$$

Without loss of generality, it can be assumed that these bases are sets of eigenvectors of noncommuting operators $\hat{X}$ and $\hat{P}$:

$$\hat{X}|x_k\rangle = k|x_k\rangle, \quad \hat{P}|p_l\rangle = l|p_l\rangle, \tag{2.2}$$

that is,

$$\hat{X} = \sum_{k=0}^{N-1} k|x_k\rangle\langle x_k|,$$

$$\hat{P} = \sum_{l=0}^{N-1} l|p_l\rangle\langle p_l|. \tag{2.3}$$

For instance, we can assume that the operators $\hat{X}$ and $\hat{P}$ are related to a discrete position and momentum of a particle on a ring with a finite number of equidistant sites [24]. Specifically, we can introduce a length scale L and two operators, the position $\hat{x}$ and the momentum $\hat{p}$, such that

$$\hat{x}|x_k\rangle = x_k|x_k\rangle, \quad \hat{p}|p_l\rangle = p_l|p_l\rangle, \tag{2.4}$$

where

$$x_k = L\sqrt{\frac{2\pi}{N}}k, \quad p_l = \frac{\hbar}{L}\sqrt{\frac{2\pi}{N}}l. \tag{2.5}$$

The length L can, for example, be taken equal to $\sqrt{\hbar/\omega m}$, where m is the mass and ω the frequency of a quantum ''harmonic''oscillator within a finite-dimensional Fock space (in what follows we use units such that $\hbar=1$).

The squared absolute values of the scalar product of eigenkets (2.2) do not depend on the indices k,l,

$$|\langle x_k|p_l\rangle|^2 = 1/N, \tag{2.6}$$

which means that pairs (k,l) form a discrete phase space [i.e., pairs (k,l) represent ''points'' of the discrete phase space] on which a Wigner function can be defined [25]. Next we introduce operators that shift (cyclically permute) the basis vectors [26]:

$$\hat{R}_x(n)|x_k\rangle = |x_{(k+n)\bmod N}\rangle,$$

$$\hat{R}_p(m)|p_l\rangle = |p_{(l+m)\bmod N}\rangle, \tag{2.7}$$

where the sums of indices are taken modulo N (this summation rule is considered throughout this paper; where it is clear we will not explicitly write the symbol mod N). The operators $\hat{R}_x(n)$ and $\hat{R}_p(m)$ can be expressed as powers of the operators $\hat{R}_x(1)$ and $\hat{R}_p(1)$, respectively,

$$\hat{R}_x(n) = \hat{R}_x^n(1), \quad \hat{R}_p(m) = \hat{R}_p^m(1). \tag{2.8}$$

In the x basis these operators can be expressed as

$$\langle x_k|\hat{R}_x(n)|x_l\rangle = \delta_{k+n,l},$$

$$\langle x_k|\hat{R}_p(m)|x_l\rangle = \delta_{k,l}\exp\left(i\frac{2\pi}{N}ml\right). \tag{2.9}$$

Moreover these operators fulfill the Weyl commutation relation [27–29]

$$\hat{R}_x(n)\hat{R}_p(m)=\exp\left(i\frac{2\pi}{N}mn\right)\hat{R}_p(m)\hat{R}_x(n); \quad (2.10)$$

although they do not commute, they form a representation of an Abelian group in a ray space. We can displace a state in arbitrary order using $\hat{R}_x(n)\hat{R}_p(m)$ or $\hat{R}_p(m)\hat{R}_x(n)$; the resulting state will be the same—the corresponding kets will differ only by an unimportant multiplicative factor. We see that the operators $\hat{R}_x(n)$ and $\hat{R}_p(m)$ displace states in the directions x and p, respectively. The product $\hat{R}_x(n)\hat{R}_p(m)$ acts as a displacement operator in the discrete phase space (k,l) [30]. These operators can be expressed via the generators of translations (shifts)

$$\hat{R}_x(n)=\exp\left(-i\frac{2\pi}{N}n\hat{P}\right)=\exp(-ix_n\hat{p}),$$

$$(2.11)$$

$$\hat{R}_p(m)=\exp\left(i\frac{2\pi}{N}m\hat{X}\right)=\exp(ip_m\hat{x}).$$

We note that the structure of the group associated with the operators $\hat{R}_x(n)$ and $\hat{R}_p(m)$ is reminiscent of the group of phase-space translations (i.e., the Heisenberg group) in quantum mechanics [31].

A general single-particle state in the x basis can be expressed as

$$|\Psi\rangle_1=\sum_{k=0}^{N-1}c_k|x_k\rangle_1, \quad \sum_{k=0}^{N-1}|c_k|^2=1. \quad (2.12)$$

The basis of maximally entangled two-particle states (the analog of the Bell basis for spin-$\frac{1}{2}$ particles) can be written as [31]

$$|\Xi_{mn}\rangle=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\exp\left(i\frac{2\pi}{N}mk\right)|x_k\rangle|x_{(k-n)\,\mathrm{mod}\,N}\rangle,$$

$$(2.13)$$

where $m,n=0,\ldots,N-1$. We can also rewrite these maximally entangled states in the p basis:

$$|\Xi_{mn}\rangle=\frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\exp\left(-i\frac{2\pi}{N}nl\right)|p_{(m-l)\,\mathrm{mod}\,N}\rangle|p_l\rangle.$$

$$(2.14)$$

The states $|\Xi_{mn}\rangle$ form an orthonormal basis

$$\langle\Xi_{kl}|\Xi_{mn}\rangle=\delta_{k,m}\delta_{l,n}, \quad (2.15)$$

with

$$\sum_{m,n=0}^{N-1}|\Xi_{mn}\rangle\langle\Xi_{mn}|=\hat{1}\otimes\hat{1}. \quad (2.16)$$

In order to prove the above relations we have used the standard relation $\Sigma_{n=0}^{N-1}\exp[2\pi i(k-k')n/N]=N\delta_{k,k'}$.

Let us denote the particles, or subsystems, in the state $|\Xi_{mn}\rangle$ as particles 2 and 3 (particle 1 will be the one we are trying to clone), and indicate this by subscripts on the state. It is then interesting to note that the whole set of N^2 maximally entangled states $|\Xi_{mn}\rangle_{23}$ can be generated from the state $|\Xi_{00}\rangle_{23}$ by the action of *local* unitary operations (shifts) of the form

$$|\Xi_{mn}\rangle_{23}=\hat{1}_2\otimes\hat{R}_x(n)\hat{R}_p(m)|\Xi_{00}\rangle_{23}, \quad (2.17)$$

acting just on system 3 in this particular case.

From the definition of the states $|\Xi_{mn}\rangle_{23}$ it follows that they are simultaneously eigenstates of the operators $\hat{X}_2-\hat{X}_3$ and $\hat{P}_2+\hat{P}_3$:

$$(\hat{X}_2-\hat{X}_3)|\Xi_{mn}\rangle_{23}=n|\Xi_{mn}\rangle_{23},$$

$$(2.18)$$

$$(\hat{P}_2+\hat{P}_3)|\Xi_{mn}\rangle_{23}=m|\Xi_{mn}\rangle_{23}.$$

We easily see that for $N=2$ the above formalism reduces to the well-known spin-$\frac{1}{2}$ particle (qubit) case.

Now we introduce generalizations of the two-qubit C-NOT gate (see also Ref. [32]). In the case of qubits the C-NOT gate is represented by a two-particle operator such that if the first (control) particle labeled a is in the state $|0\rangle$ nothing "happens" to the state of the second (target) particle labeled b. If, however, the control particle is in the state $|1\rangle$ then the state of the target is "flipped," i.e., the state $|0\rangle$ is changed into the state $|1\rangle$ and vice versa. Formally we can express the action of this C-NOT gate as a two-qubit operator of the form

$$\hat{D}_{ab}=\sum_{k,m=0}^{1}|k\rangle_a\langle k|\otimes|(m+k)\,\mathrm{mod}\,2\rangle_b\langle m|. \quad (2.19)$$

We note that in principle one can introduce an operator $\hat{D}_{ab}^\dagger$ defined as

$$\hat{D}_{ab}^\dagger=\sum_{k,m=0}^{1}|k\rangle_a\langle k|\otimes|(m-k)\,\mathrm{mod}\,2\rangle_b\langle m|. \quad (2.20)$$

In the case of qubits these two operators are equal. This is not the case when the dimension of the Hilbert space is larger than 2 [32]. Let us generalize the above definition of the operator $\hat{D}$ for $N>2$. Before doing so, we shall simplify our notation. Because we will work mostly in the x basis we shall use the notation $|x_k\rangle\equiv|k\rangle$ where it may be done so unambiguously. With this in mind we now write

$$\hat{D}_{ab}=\sum_{k,m=0}^{N-1}|k\rangle_a\langle k|\otimes|(m+k)\,\mathrm{mod}\,N\rangle_b\langle m|. \quad (2.21)$$

From the definition (2.21) it follows that the operator $\hat{D}_{ab}$ acts on the basis vectors as

$$\hat{D}_{ab}|k\rangle|m\rangle=|k\rangle|(k+m)\,\mathrm{mod}\,N\rangle, \quad (2.22)$$
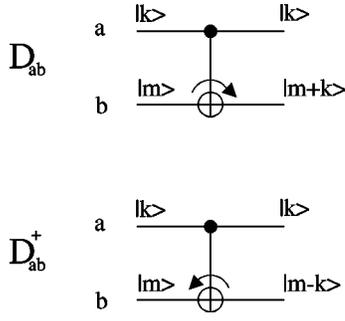
FIG. 1. Schematic description of the action of the control shift gates $\hat{D}_{ab}$ and $\hat{D}_{ab}^{\dagger}$ as given by Eqs. (2.23) and (2.24).

which means that this operator is equal to the conditional adder [33,34] and can be performed with the help of a simple quantum network as discussed in [33].

If we take into account the definition of the shift operator $\hat{R}_x(n)$ given by Eq. (2.7) and the definition of the position and momentum operators $\hat{x}$ and $\hat{p}$ given by Eq. (2.8) we can rewrite the operator $\hat{D}_{ab}$ as

$$\hat{D}_{ab} = \sum_{k,m=0}^{N-1} |k\rangle_a\langle k| \otimes \hat{R}_x^{(b)}(k)|m\rangle_b\langle m| \equiv \sum_{k=0}^{N-1} |k\rangle_a\langle k|$$

$$\otimes \hat{R}_x^{(b)}(k) = e^{-i\hat{x}_a\hat{p}_b}, \qquad (2.23)$$

and analogously

$$\hat{D}_{ab}^{\dagger} = \sum_{k,m=0}^{N-1} |k\rangle_a\langle k| \otimes |(m-k)\bmod N\rangle_b\langle m| \equiv \sum_{k=0}^{N-1} |k\rangle_a\langle k|$$

$$\otimes \hat{R}_x^{(b)}(-k) = e^{i\hat{x}_a\hat{p}_b}, \qquad (2.24)$$

where the superscripts a and b indicate on which Hilbert space the given operator acts. Now we see that for $N>2$ the two operators $\hat{D}$ and $\hat{D}^{\dagger}$ do differ; they describe conditional shifts in opposite directions. We see that the generalization of the C-NOT operator are the *conditional shifts*. The amount by which the target (in our case particle b) is shifted depends on the state of the control particle (a). In Fig. 1 we show a graphical representation of these conditional shift gates.

Continuous limit

In the $N\rightarrow\infty$ limit we have to take special care in handling the expressions for the eigenstates of the position and momentum operators [35]. To avoid divergences we have to regularize our states by ''smearing'' them. In other words, the eigenstate of the operator $\hat{x}$ is replaced by a squeezed displaced state (see, e.g., Ref. [30]) with reduced quadrature fluctuations in the $\hat{x}$ direction (see below). To express these states explicitly we utilize the Wigner function representation, which for pure states is defined as

$$W_{|\Psi\rangle}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, \varphi_{|\Psi\rangle}(x_-)\varphi_{|\Psi\rangle}^*(x_+)\, e^{ipz},$$

$$(2.25)$$

where $x_{\pm} = (x \pm z/2)$ and $\varphi_{|\Psi\rangle}(x)$ is the wave function of the state $|\Psi\rangle$, i.e., $\varphi_{|\Psi\rangle}(x) = \langle x|\Psi\rangle$, which in the $|x\rangle$ basis is expressed as

$$|\Psi\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \varphi_{|\Psi\rangle}(x)|x\rangle, \qquad (2.26)$$

where we have used the relation $\langle x|y\rangle = \sqrt{2\pi}\,\delta(x-y)$. The Wigner function $W_{|\Psi\rangle}(x,p)$ is a quasiprobability distribution in phase space and is normalized so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, dp\, W_{|\Psi\rangle}(x,p) = 1, \qquad (2.27)$$

where $dx\, dp/2\pi$ is the invariant measure in phase space (here we have taken $\hbar = 1$).

With these definitions we can represent a regularized version of an eigenstate of the position operator $\hat{x}$ with mean value equal to zero, $|x_0\rangle$, as a state described by the Gaussian Wigner function

$$W_{|x_0\rangle}(x,p) = 2\exp(-e^{2\xi}x^2 - e^{-2\xi}p^2), \qquad (2.28)$$

for which the variances of the position and momentum operators are $(\Delta\hat{x})^2 = \frac{1}{2}e^{-2\xi}$ and $(\Delta\hat{p})^2 = \frac{1}{2}e^{2\xi}$, respectively. The state (2.28) is a minimum uncertainty state, i.e., $(\Delta\hat{x})(\Delta\hat{p}) = \frac{1}{2}$ irrespective of the value of the squeezing parameter ξ. For the mean excitation number we find the expression $\bar{n} = \sinh^2\xi$. We see that in the limit $\xi\rightarrow\infty$, the state described by Wigner function (2.28) is indeed a state with no fluctuations in the x direction at the expense of infinite fluctuations in the p direction. In other words in the limit $\xi \rightarrow\infty$ the state (2.28) is an eigenstate of $\hat{x}$.

Analogously a regularized eigenstate of the momentum operator is described by the Wigner function

$$W_{|p_0\rangle}(x,p) = 2\exp(-e^{-2\xi}x^2 - e^{2\xi}p^2), \qquad (2.29)$$

where for the variances of the position and momentum operators we find $(\Delta\hat{x})^2 = \frac{1}{2}e^{2\xi}$ and $(\Delta\hat{p})^2 = \frac{1}{2}e^{-2\xi}$, respectively.

The wave functions corresponding to the states (2.28) and (2.29) read

$$\varphi_{|x_0\rangle}(x) = 2^{1/4}e^{\xi/2}\exp\left(-\frac{e^{2\xi}x^2}{2}\right) \qquad (2.30)$$

and

$$\varphi_{|p_0\rangle}(x) = 2^{1/4}e^{-\xi/2}\exp\left(-\frac{e^{-2\xi}x^2}{2}\right), \qquad (2.31)$$

respectively. We denote the corresponding ket vectors as $|x_0(\xi)\rangle$ and $|p_0(\xi)\rangle$, where we have explicitly indicated that these states are regularized versions of two specific eigenstates of the position and momentum operators. The x distribution of the state $\varphi_{|x_0\rangle}(x)$ is defined as usual, i.e., $P_{|x_0\rangle}(x)$

$=|\varphi_{|x_0\rangle}(x)|^2$, and is normalized to unity as $\int dx\, P_{|x_0\rangle}(x)/\sqrt{2\pi}$ $=1$. This distribution can also be obtained from the Wigner function (2.28) via integration over p, i.e.,

$$P_{|x_0\rangle}(x)=\frac{1}{\sqrt{2\pi}}\int dp\, W_{|x_0\rangle}(x,p)=2^{1/2}e^{\xi}\exp(-e^{2\xi}x^2),$$
(2.32)

which in the large ξ limit gives $\lim_{\xi\to\infty}P_{|x_0\rangle}(x)=\sqrt{2\pi}\,\delta(x)$, as expected.

In an analogous way we define a maximally entangled two-mode state $|\Xi_{00}(\xi)\rangle$ in the continuous limit. Specifically, we define this state in a regularized form for which the Wigner function reads [36]

$$W_{|\Xi_{00}\rangle}(x_1,p_1;x_2,p_2)=4\exp\left\{-\frac{e^{2\xi}}{2}[(x_1-x_2)^2+(p_1+p_2)^2]\right.$$
$$\left.-\frac{e^{-2\xi}}{2}[(x_1+x_2)^2+(p_1-p_2)^2]\right\}.$$
(2.33)

This is a Wigner function describing a two-mode squeezed vacuum. If we trace over one of the modes, i.e., if we perform an integration over the parameters x_2 and p_2, we obtain from Eq. (2.33) a Wigner function of a thermal field,

$$W_{th}(x_1,p_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_2\,dp_2 W_{|\Xi_{00}\rangle}(x_1,p_1;x_2,p_2)$$
$$=\frac{2}{1+2\bar{n}}\exp\left(-\frac{x_1^2+p_1^2}{1+2\bar{n}}\right),$$
(2.34)

where $\bar{n}=\sinh^2\xi$ is the mean excitation number in the two-mode squeezed vacuum under consideration. We note that the thermal state (2.34) is a maximally mixed state (i.e., with the state with the highest value of the von Neumann entropy) for a given mean excitation number. This means that the pure state (2.33) is the most entangled state for a given mean excitation number. From this it follows that to create a truly maximally entangled state, i.e., the state (2.33) in the limit $\xi\to\infty$, an infinite number of quanta is needed and so infinite energy.

The two-mode wave function of the state (2.33) in the x representation reads

$$\varphi_{|\Xi_{00}\rangle}(x_1;x_2)=\sqrt{2}\exp\left(-\frac{e^{2\xi}}{4}\tilde{x}_-^2-\frac{e^{-2\xi}}{4}\tilde{x}_+^2\right),\quad(2.35)$$

where $\tilde{x}_\pm=(x_1\pm x_2)$. In what follows we shall denote this regularized version of the maximally entangled state in a semi-infinite Hilbert space as $|\Xi_{00}(\xi)\rangle$. Now that we have laid out the formalism, we can resume our discussion.
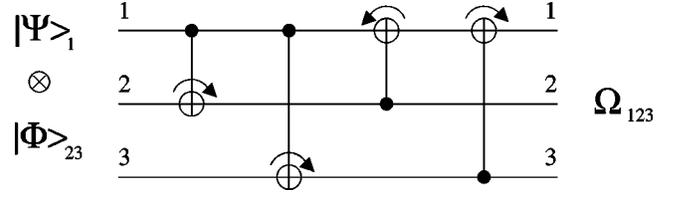

FIG. 2. A logic network of the QID as given by the unitary transformation (3.4).

III. NETWORK FOR QUANTUM-INFORMATION DISTRIBUTION

We have shown earlier [3,12,37] that to perform quantum cloning we need, apart from systems 1 and 2, between which the information is shared, an additional quantum system 3 that mediates the distribution of the quantum-information. Following this philosophy, we assume a quantum information distributor to be a two-particle system (2 and 3) each of the same physical type as the original system 1. Let us assume that the quantum distributor is initially prepared in the most general two-particle pure state

$$|\Phi\rangle_{23}=\sum_{m,k=0}^{N-1}d_{mk}|m\rangle_2|k\rangle_3.$$
(3.1)

In analogy with the quantum computational network used in the quantum cloner [37] we assume the QID network to be

$$\hat{U}_{123}=\hat{D}_{31}\hat{D}_{21}^{\dagger}\hat{D}_{13}\hat{D}_{12},$$
(3.2)

with the idea being that the flow of information in the quantum distributor, as described by the unitary operator (3.2), is governed by the preparation of the distributor itself, i.e., by the choice of the state (3.1). In other words, we imagine the transformation (3.2) as a universal ''processor'' or distributor and the state (3.1) as ''software'' through which the information flow is controlled. Using relation (2.23) we can rewrite the QID transformation as (see also Ref. [38])

$$\hat{U}_{123}=\exp[-i(\hat{x}_3-\hat{x}_2)\hat{p}_1]\exp[-i\hat{x}_1(\hat{p}_2+\hat{p}_3)].\quad(3.3)$$

The distribution of information encoded in the original particle is performed via a sequence of four conditional shifts $\hat{D}$. The output state of the three-particle system after the four controlled shifts are applied is

$$|\Omega\rangle_{123}=\hat{D}_{31}\hat{D}_{21}^{\dagger}\hat{D}_{13}\hat{D}_{12}|\Psi\rangle_1|\Phi\rangle_{23}.$$
(3.4)

We present the graphical representation of the logical network (3.4) with the conditional shift gates $\hat{D}_{ab}$ in Fig. 2.

The four operators $\hat{D}$ act on the basis vectors $|n\rangle_1|m\rangle_2|k\rangle_3$ as

$$\hat{D}_{31}\hat{D}_{21}^{\dagger}\hat{D}_{13}\hat{D}_{12}|n\rangle_1|m\rangle_2|k\rangle_3$$
$$=|(n-m+k)\bmod N\rangle_1\,|(m+n)\bmod N\rangle_2$$
$$\times|(k+n)\bmod N\rangle_3.$$
(3.5)

As we shall see, the choice of the state $|\Phi\rangle_{23}$ controls the flow of the quantum information contained in the state $|\Psi\rangle_1$ through the QID.

Before examining this issue, however, it is useful to explore the covariance properties of this distributor for any choice of $|\Phi\rangle_{23}$. A device is covariant with respect to the transformation $\hat{U}$, if application of $\hat{U}$ to the input, i.e., $|\Psi\rangle \rightarrow \hat{U}|\Psi\rangle$, implies that the output density matrix representing the pair of outputs, $\hat{\rho}^{(\text{out})}$, transforms as [39]

$$\rho^{(\text{out})} \rightarrow \hat{U} \otimes \hat{U} \rho^{(\text{out})} \hat{U}^{-1} \otimes \hat{U}^{-1}. \tag{3.6}$$

When examining whether the distributor is covariant with respect to transformations of the form $\hat{R}_x(n)\hat{R}_p(n)$, it is sufficient to confirm this covariant action for ''displacements'' along the x and p axes separately, given by $\hat{R}_x(n)$ and $\hat{R}_p(n)$, respectively. If the state to be distributed is $\hat{R}_x(n)|\Psi\rangle_1$ we find

$$\hat{R}_x(n)_1|\Psi\rangle_1|\Phi\rangle_{23} \rightarrow \hat{R}_x(n)_1\hat{R}_x(n)_2\hat{R}_x(n)_3|\Omega\rangle_{123}, \tag{3.7}$$

where $|\Omega\rangle_{123}$ is given by Eq. (3.4). Similarly, if the input state is $\hat{R}_p(n)|\Psi\rangle_1$, we have

$$\hat{R}_p(n)_1|\Psi\rangle_1|\Phi\rangle_{23} \rightarrow \hat{R}_p(n)_1\hat{R}_p(n)_2\hat{R}_p(-n)_3|\Omega\rangle_{123}. \tag{3.8}$$

Combinations of these two ''displacements'' act in the natural way, so that if we ''translate'' the input state by a certain amount, the reduced density matrixes of the three outputs are translated by the same amount, and if we perform a momentum ''translation'' on the input state, the reduced density matrices of outputs 1 and 2 are translated in momentum by the same amount, while that of output 3 has its momentum translated by the opposite amount. This implies that this QID is covariant with respect to translations and momentum translations, and that the fidelities of the output reduced density matrices are unaffected when these transformations are applied to the input.

Having established the covariant action of our distributor in arbitrary dimensions for any input state $|\Phi\rangle_{23}$, we now wish to determine how this state affects the flow of quantum information in the QID.

(i) Let us first assume that the QID state $|\Phi\rangle$ is initially prepared in the maximally entangled state $|\Xi_{00}\rangle_{23}$ given by Eq. (2.14). Taking the original system to be prepared in the state (1.1) we find after the QID transformation

$$\hat{U}_{123}|\Psi\rangle_1|\Xi_{00}\rangle_{23} = |\Psi\rangle_1|\Xi_{00}\rangle_{23}, \tag{3.9}$$

that system 1 remains in the original state while the QID remains in its initial maximally entangled state $|\Xi_{00}\rangle_{23}$. This means that even though the three-particle system has interacted via four controlled shifts the total state is unchanged.

(ii) Instead, let us assume the QID is initially prepared in the product state

$$|\Phi\rangle_{23} = |x_0\rangle_2|p_0\rangle_3, \tag{3.10}$$

where $|p_0\rangle$ is an eigenstate of the momentum operator with the mean value equal to zero. At the output we then find

$$\hat{U}_{123}|\Psi\rangle_1|x_0\rangle_2|p_0\rangle_3 = |\Psi\rangle_2|\Xi_{00}\rangle_{13}, \tag{3.11}$$

which means that the information from the system 1 is completely transferred to the system 2 while at the output the systems 1 and 3 are in the maximally entangled state $|\Xi_{00}\rangle_{13}$. Note that the output here is a state-exchanged version of the output in Eq. (3.9).

Since these two cases realize the two extreme situations (no information transfer and complete information transfer) it is natural to ask what is the action of the QID if it is prepared in a linear superposition of the states $|\Xi_{00}\rangle_{23}$ and $|x_0\rangle_2|p_0\rangle_3$. Let us take the input state of the QID to be

$$|\Phi\rangle_{23} = \alpha|\Xi_{00}\rangle_{23} + \beta|x_0\rangle_2|p_0\rangle_3, \tag{3.12}$$

where α and β are real parameters. Note that from the normalization condition $\langle\Phi|\Phi\rangle = 1$ it follows that these parameters must fulfill the condition

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{N} = 1. \tag{3.13}$$

When the QID transformation is applied with the QID initially prepared in the state (3.12) the output state becomes

$$\hat{U}_{123}|\Psi\rangle_1|\Phi\rangle_{23} = \alpha|\Psi\rangle_1|\Xi_{00}\rangle_{23} + \beta|\Psi\rangle_2|\Xi_{00}\rangle_{13}. \tag{3.14}$$

Tracing over the systems 2 (1) and 3 we find the reduced stated for system 1 (2) at the output to be described by

$$\hat{\rho}_1^{(\text{out})} = \left(\alpha^2 + \frac{2\alpha\beta}{N}\right)\hat{\rho}^{(\text{in})} + \frac{\beta^2}{N}\hat{1},$$

$$\hat{\rho}_2^{(\text{out})} = \left(\beta^2 + \frac{2\alpha\beta}{N}\right)\hat{\rho}^{(\text{in})} + \frac{\alpha^2}{N}\hat{1}, \tag{3.15}$$

$$\hat{\rho}_3^{(\text{out})} = \frac{2\alpha\beta}{N}(\hat{\rho}^{(\text{in})})^{\text{T}} + \frac{(N - 2\alpha\beta)}{N^2}\hat{1},$$

where $\hat{\rho}^{(\text{in})} = |\Psi\rangle\langle\Psi|$ is the density operator of the original state of system 1, and $(\hat{\rho})^{\text{T}}$ is the transposed operator.

Taking into account condition (3.13) we can directly rewrite the last two density operators in the form (1.2). This means that QID is the covariant transformation that in a controlled way distributes information between the two systems. There is a price to pay for this covariant information distribution which is reflected by the additional noise.

Cloner

Let us assume that $\alpha = \beta$, i.e., the two outputs (3.15) are equal. In this case QID acts as a universal quantum cloner for arbitrary dimensions. From Eq. (3.12) we find the initial state of the cloner to be

$$|\Phi\rangle_{23} = \frac{1}{\sqrt{2(N+1)}} \sum_{m=0}^{N-1} (|x_0\rangle_2 + |x_m\rangle_2)|x_m\rangle_3. \quad (3.16)$$

With this initial QID state the output of the cloner yields two clones of the form

$$\hat{\rho}_j^{(\text{out})} = s\hat{\rho}_j^{(\text{in})} + \frac{1-s}{N}\hat{\mathbb{1}}, \quad j=1,2. \quad (3.17)$$

The scaling factor s is

$$s = \frac{N+2}{2(N+1)}. \quad (3.18)$$

Finally, system 3 of the cloner has a reduced state given by

$$\hat{\rho}_3^{(\text{out})} = \frac{1}{N+1}(\hat{\rho}^{(\text{in})})^{\text{T}} + \frac{1}{N+1}\hat{\mathbb{1}}, \quad (3.19)$$

i.e., this piece of the cloner is left in a state proportional to the transposed state of the original quantum system plus completely random noise.

IV. CONTINUOUS LIMIT

In what follows we make a connection between the discrete and the continuous cases. The role of the controlled shifts (NOT's) in the continuous limit is obvious—it is a conditional shift down the x axis in phase space. Consequently, the QID operator (3.3) has a clear meaning in the continuous limit. Our goal now is to find the continuous analog of the initial state $|\Phi\rangle_{23}$ (3.12) of the QID. This is rather straightforward: we simply need to use the regularized versions of the states $|x_0(\xi)\rangle$, $|p_0(\xi)\rangle$, and $|\Xi_{00}(\xi)\rangle$ as introduced in Sec. II. The input state of the QID in the continuous case can then be written as

$$|\Phi(\xi)\rangle_{23} = \alpha|\Xi_{00}(\xi)\rangle_{23} + \beta|x_0(\xi)\rangle_2|p_0(\xi)\rangle_3, \quad (4.1)$$

which in the x basis becomes

$$|\Phi(\xi)\rangle_{23} = \frac{1}{2\pi}\int dx_2\, dx_3\, \mu(x_2,x_3)|x_2\rangle|x_3\rangle, \quad (4.2)$$

where

$$\mu(x_2,x_3) = \alpha\psi_{|\Xi_{00}\rangle}(x_2,x_3) + \beta\psi_{|x_0\rangle}(x_2)\psi_{|p_0\rangle}(x_3), \quad (4.3)$$

and the Gaussian functions $\psi_{|x_0\rangle}(x_2)$, $\psi_{|p_0\rangle}(x_3)$, and $\psi_{|\Xi_{00}\rangle}(x_2,x_3)$ are defined by Eqs. (2.30), (2.31), and (2.35), respectively.

For finite values of squeezing the states $|\Xi_{00}(\xi)\rangle_{23}$ and $|x_0(\xi)\rangle_2|p_0(\xi)\rangle_3$ are not mutually orthogonal, and therefore, in order to fulfill the normalization condition for the state $|\Phi\rangle$, the parameters α and β have to fulfill a condition analogous to Eq. (3.13),

$$\alpha^2 + \beta^2 + \frac{4\alpha\beta}{\sqrt{4+2\sinh^2 2\xi}} = 1. \quad (4.4)$$

With this initial preparation of the QID, the universal (covariant) information distribution of continuous variables is realized using the network described in Eq. (3.2). The operator $\hat{U}_{123}$ acts on the basis states $|x_1\rangle_1|x_2\rangle_2|x_3\rangle_3$ as

$$\hat{U}_{123}|x_1\rangle_1|x_2\rangle_2|x_3\rangle_3 = |z_1\rangle_1|z_2\rangle_2|z_3\rangle_3, \quad (4.5)$$

where $z_1 = x_1 - x_2 + x_3$, $z_2 = x_1 + x_2$, and $z_3 = x_1 + x_3$. Assuming that the original system is initially prepared in the state

$$|\Psi\rangle_1 = \frac{1}{\sqrt{2\pi}}\int dx_1\, \psi(x_1)|x_1\rangle_1, \quad (4.6)$$

the output of the QID becomes

$$\hat{U}_{123}|\Psi\rangle_1|\Phi(\xi)\rangle_{23} = \frac{1}{(2\pi)^{3/2}}\int dx_1\, dx_2\, dx_3$$
$$\times \psi(x_1)\mu(x_2,x_3)\,|x_1-x_2+x_3\rangle_1$$
$$\times |x_1+x_2\rangle_2\,|x_1+x_3\rangle_3. \quad (4.7)$$

Upon tracing out modes 2 and 3 we obtain from Eq. (4.7) the density operator describing the original system at the output of the QID:

$$\hat{\rho}_1^{(\text{out})} = \frac{1}{(2\pi)^{3/2}}\int d\eta\, dx_1\, dx_1'\, \psi(x_1)\psi^*(x_1')\mathcal{K}(x_1-x_1';\eta)$$
$$\times |x_1+\eta\rangle\langle x_1'+\eta|, \quad (4.8)$$

where the integral kernel $\mathcal{K}(x_1-x_1';\eta)$ is given by the expression

$$\mathcal{K}(x_1-x_1';\eta) = \frac{1}{2\sqrt{2\pi}}\int d\chi\, \mu\left(\frac{\chi-\eta}{2}-x_1;\frac{\chi+\eta}{2}-x_1\right)$$
$$\times \mu\left(\frac{\chi-\eta}{2}-x_1';\frac{\chi+\eta}{2}-x_1'\right). \quad (4.9)$$

From the fact that the trace of the density matrix (4.8) is equal to unity we find that the integral kernel (4.9) has to fulfill the condition

$$\frac{1}{\sqrt{2\pi}}\int d\eta\, \mathcal{K}(0;\eta) = 1. \quad (4.10)$$

The kernel itself can be expressed in the form

$$\mathcal{K}(\bar{x}_1;\eta) = \alpha^2\mathcal{K}_1(\bar{x}_1;\eta) + \beta^2\mathcal{K}_2(\bar{x}_1;\eta) + \alpha\beta\mathcal{K}_3(\bar{x}_1;\eta), \quad (4.11)$$

where we have introduced the notation $\bar{x}_1 = x_1 - x_1'$. Using the explicit expressions for the wave functions describing the input state of the QID we find for the kernel functions

$$\mathcal{K}_1(\bar{x}_1;\eta) = e^{\xi}\exp\left(-\frac{e^{-2\xi}}{2}\bar{x}_1^2 - \frac{e^{2\xi}}{2}\eta^2\right), \quad (4.12)$$

$$\mathcal{K}_2(\bar{x}_1;\eta) = \frac{1}{\sqrt{\cosh 2\xi}}\exp\left(-\frac{\cosh 2\xi}{2}\bar{x}_1^2 - \frac{1}{2\cosh 2\xi}\eta^2\right), \quad (4.13)$$

$$\mathcal{K}_3(\bar{x}_1;\eta) = \frac{2}{\sqrt{3e^{-2\xi}+e^{2\xi}}}$$

$$\times \exp\left[-\frac{e^{-4\xi}(1+e^{4\xi})\bar{x}_1^2 + (2+\sinh^2 2\xi)\eta^2}{3e^{-2\xi}+e^{2\xi}}\right]$$

$$\times\left\{\exp\left[-\frac{e^{-4\xi}(1-e^{4\xi})\bar{x}_1}{3e^{-2\xi}+e^{2\xi}}\right]\right.$$

$$\left.+\exp\left[-\frac{e^{-4\xi}(-1+e^{4\xi})\bar{x}_1}{3e^{-2\xi}+e^{2\xi}}\right]\right\}. \quad (4.14)$$

It is now easy to check that

$$\frac{1}{\sqrt{2\pi}}\int d\eta\,\mathcal{K}_1(0;\eta) = \frac{1}{\sqrt{2\pi}}\int d\eta\,\mathcal{K}_2(0;\eta) = 1,$$

$$\frac{1}{\sqrt{2\pi}}\int d\eta\,\mathcal{K}_3(0;\eta) = \frac{4}{\sqrt{4+2\sinh^2 2\xi}}, \quad (4.15)$$

from which it follows that the kernel $\mathcal{K}(0;\eta)$ satisfies condition (4.10).

In what follows we utilize the Wigner-function formalism to analyze the performance of the cloning machine. We find a Wigner $W(x,p)_1^{(\text{out})}$ of the output state (4.8) which we express as a convolution of the Wigner function $W_1^{(\text{in})}(x,p)$ of the input mode and the Wigner function $W^{\mathcal{K}}(x,p)$ of the kernel (4.11):

$$W(x,p)_1^{(\text{out})} = \frac{1}{2\pi}\int dx'\,dp'\,W^{\mathcal{K}}(x',p'),$$

$$\times W_1^{(\text{in})}(x-x',p+p'), \quad (4.16)$$

where

$$W^{\mathcal{K}}(x',p') = \frac{1}{\sqrt{2\pi}}\int dz\,e^{ip'z}\mathcal{K}(z,x'). \quad (4.17)$$

From our definitions it follows that

$$W^{\mathcal{K}}(x',p') = \alpha^2 W^{\mathcal{K}_1}(x',p') + \beta^2 W^{\mathcal{K}_2}(x',p')$$

$$+ \alpha\beta W^{\mathcal{K}_3}(x',p'), \quad (4.18)$$

where $W^{\mathcal{K}_j}(x',p')$ are the Wigner functions of the kernels $\mathcal{K}_j$ ($j=1,2,3$). We can easily check that

$$\frac{1}{2\pi}\int dx'\,dp'\,W^{\mathcal{K}}(x',p') = 1 \quad (4.19)$$

which is equivalent to the condition (4.10).

From Eq. (4.12) we find the Wigner function $W^{\mathcal{K}_1}(x',p')$

$$W^{\mathcal{K}_1}(x',p') = e^{2\xi}\exp\left[-\frac{e^{2\xi}}{2}(x'^2+p'^2)\right], \quad (4.20)$$

which in the limit of large squeezing reads

$$W^{\mathcal{K}_1}(x',p') \rightarrow 2\pi\,\delta(x')\,\delta(p'). \quad (4.21)$$

The Wigner function $W^{\mathcal{K}_2}(x',p')$ of the kernel (4.13) reads

$$W^{\mathcal{K}_2}(x',p') = \frac{1}{1+2\bar{n}}\exp\left[-\frac{(x'^2+p'^2)}{2(1+2\bar{n})}\right], \quad (4.22)$$

where we have used the notation $\bar{n}=\sinh^2\xi$, so that $\cosh 2\xi = 1+2\bar{n}$. We note that this Wigner function for large squeezing (i.e., the large $\bar{n}$ limit) is equal to the Wigner function of a *thermal* state (2.34) with the mean number of excitations equal to $2\bar{n}$.

Analogously we can evaluate the explicit expression for the Wigner function $W^{\mathcal{K}_3}(x',p')$ of the kernel (4.15). This is rather cumbersome, and, since we are interested only in the large squeezing limit, we present the corresponding Wigner function only in this limit:

$$W^{\mathcal{K}_3}(x',p') \simeq 2\sqrt{2}\exp\left[-\frac{e^{2\xi}}{4}(x'^2+p'^2)\right], \quad (4.23)$$

which in the large ξ limit can be formally expressed as

$$W^{\mathcal{K}_3}(x',p') \rightarrow 8\sqrt{2}\pi e^{-2\xi}\delta(x')\,\delta(p'). \quad (4.24)$$

Now we can give the explicit expression for the Wigner function $W_1^{(\text{out})}(x,p)$ of the output mode 1, for which we find

$$W_1^{(\text{out})}(x,p) = \alpha^2 W_1^{(\text{in})}(x,p) + \frac{\beta^2}{2\pi}\int dx'\,dp'\,W_1^{(\text{in})}$$

$$\times(x-x',p-p')W^{\mathcal{K}_2}(x',p')$$

$$+4\sqrt{2}e^{-2\xi}\alpha\beta W_1^{(\text{in})}(x,p). \quad (4.25)$$

We note that in the large ξ limit the third term in right-hand side of Eq. (4.25) will vanish due to the factor $e^{-2\xi}$. Taking into account that in the large squeezing (i.e., large $\bar{n}$) limit the function $W^{\mathcal{K}_2}(x,p)$ is essentially equal to a Wigner function of a thermal field (2.34) with a mean excitation number of $2\bar{n}$ [we will denote this Wigner function as $W_{\text{th}}(x,p;2\bar{n})$], we can rewrite the Wigner function (4.25) at the output of the QID as

$$W_1^{(\text{out})}(x,p) = \alpha^2 W_1^{(\text{in})}(x,p) + \frac{\beta^2}{2\pi}\int dx'\,dp'\,W_1^{(\text{in})}(x',p')$$

$$\times W_{\text{th}}(x-x',p-p';2\bar{n}). \quad (4.26)$$

Therefore, the output Wigner function is simply α^2 times the input Wigner function plus β^2 times the convolution of the input Wigner function and that of a thermal state.

Finally we evaluate the fidelity of the QID transformation, which is defined as

$$\mathcal{F}_j = \langle \Psi | \hat{\rho}_j^{(out)} | \Psi \rangle = \frac{1}{2\pi} \int dx\, dp\, W_1^{(in)}(x,p) W_j^{(out)}(x,p).$$
(4.27)

In the large squeezing limit the fidelity (4.27) can be approximated as

$$\int dx\, dp \int dx'\, dp'\ W^{(in)}(x,p) W_{th}(x-x', p-p', 2\bar{n})$$

$$\times W^{(in)}(x',p')$$

$$\simeq \frac{2\pi}{2\bar{n}+1} \int dx \int dx'\, |\psi(x)|^2 |\psi(x')|^2$$

$$\times e^{-(x-x')^2/2(2\bar{n}+1)}.$$
(4.28)

The integrals on the right-hand side of this equation are less than or equal to 1, so that the entire right-hand side goes to zero as $1/\bar{n}$ as $\bar{n} \to \infty$. Therefore, we find that in the large squeezing limit the fidelity of the QID is indeed input-state independent, and $\mathcal{F}_1 = \alpha^2$ while $\mathcal{F}_2 = \beta^2$.

Universal continuous cloner

If in the limit $\xi \to \infty$ we take $\alpha^2 = \beta^2 = \frac{1}{2}$ then the transformations described above describe the symmetric cloner. In the limit $\xi \to \infty$, the fidelity of the output density matrices to the original state is $\frac{1}{2}$. This is consistent with what we expect from the limit $N \to \infty$ limit of Eq. (3.17).

The fact that in the continuous case the fidelity of the copies is $\frac{1}{2}$ is suggestive; it makes one think of a coin toss. In fact, one can construct a continuous (universal) cloner, which is much simpler than the one given above but achieves the same fidelity, whose most important component is a flipping coin [13]. This cloner has two inputs, one for the state we wish to clone and one for a completely random state (ideally an infinite-temperature thermal state). What the cloner does is to flip a coin, and if the result is heads the original input state is sent to output 1 and the random state to output 2. If instead the result is tails the input state is sent to output 2 and the random state to output 1. Assuming that, on average, the overlap between the input state and the random state is small, this ''cloner'' will clone the input with a fidelity of $\frac{1}{2}$. From this we can conclude that for continuous quantum systems the universal cloner is effectively a completely classical device. Indeed, one can verify that in this limit there is no entanglement between systems 1 and 2 of the outgoing particles. This is not true in any finite-dimensional case. Taking this classical distribution as a hint, we can see that this type of continuous cloner is easily generalized to the case of an arbitrary number of inputs M_{in} and an arbitrary number of outputs M_{out} with $M_{out} \geq M_{in}$. In this case the fidelity of cloning is just M_{in}/M_{out}, which agrees

with Werner's result [17] for the optimal cloner in the infinite-dimensional limit.

V. CONCLUSIONS

We have shown that for any dimension the optimal universal quantum cloner can be constructed from essentially the same quantum circuit, i.e., what we have is a universal design for universal cloners. In the case of infinite dimensions (which includes continuous variable quantum systems) the universal cloner reduces to a classical device. By contrast, Cerf *et al.*, have shown that, if one designs a continuous cloner optimized to copy certain sets of states, then one can achieve higher fidelities than those available to the truly universal cloners studied here [38]. In particular, they showed that it is possible to design a cloner that will copy any coherent state with a fidelity of 2/3. Their cloner also fits within the structure of the QID analyzed here if one chooses the initial cloner state as

$$|\Phi\rangle_{23} = \frac{1}{\sqrt{2\pi^2}} \int dx_2 dx_3 \exp\left(-\frac{x_2^2 + x_3^2}{2}\right) |x_2\rangle |x_2 + x_3\rangle.$$
(5.1)

It is interesting to note that this cloner also produces approximate versions of the transpose of coherent states at its third output. The transpose of the coherent state $|z\rangle$ is $|z^*\rangle$. If the input to the cloner is $|z\rangle$, then $\hat{\rho}_3^{(out)}$ is a Gaussian state (that is, $\langle x | \hat{\rho}_3^{(out)} | x' \rangle$ is a Gaussian) that is concentrated about the point z^* in phase space. It is, however, more spread out than a coherent state, and its fidelity with the actual transposed state is 1/2.

For continuous systems, these specialized cloners will be more useful than the universal one. Because of their covariance properties, they will clone any two states that differ by only a translation in phase space with the same fidelity. For example, the fact that the cloner in Ref. [38] clones the vacuum with fidelity 2/3 implies that it clones all coherent states with the same fidelity. One can easily imagine generalizing this result and designing cloners to optimally clone entire classes of states; if by choosing the correct $|\Phi\rangle_{23}$ the cloner has been optimized to clone a particular state $|\Psi\rangle$, by covariance it will automatically be optimal for all states generated from $|\Psi\rangle$ by displacements in phase space. For finite-dimensional systems, however, universal cloners do better than classical devices and the simple universal circuit presented here shows how they may be constructed.

In the present paper we have concentrated on $1 \to 2$ quantum-information distributors. Using the ideas presented in Ref. [40] it is possible to design a network for the $1 \to M$ QID in any dimension. In spite of the fact that some work on $N \to M$ cloning in the continuous limit has been done (e.g., cloning of coherent states has been considered in Ref. [14]) it is still an open question how to construct quantum logic networks for an $N \to M$ QID.

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