

No-Signaling Condition and Quantum Dynamics

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We show that the basic dynamical rules of quantum physics can be derived from its static properties and the condition that superluminal communication is forbidden. More precisely, the fact that the dynamics has to be described by linear completely positive maps on density matrices is derived from the following assumptions: (1) physical states are described by rays in a Hilbert space, (2) probabilities for measurement outcomes at any given time are calculated according to the usual trace rule, and (3) superluminal communication is excluded. This result also constrains possible nonlinear modifications of quantum physics.

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The special theory of relativity is one of the cornerstones of our present scientific world view. One of its most important features is the fact that there is a maximum velocity for signals, i.e., for anything that carries information, identical to the velocity of light in vacuum. Another cornerstone of our present understanding of the world is quantum physics. Quantum physics seems to have “nonlocal” characteristics due to the existence of entanglement. Most importantly, it is not compatible with local hidden variables, as shown by the violation of Bell’s inequalities [1], which has been experimentally confirmed in many experiments [2].

It is very remarkable that, in spite of its nonlocal features, quantum mechanics is compatible with the special theory of relativity, if it is assumed that operators referring to spacelike separated regions commute. In particular, the theory obeys the “no-signaling condition”: one cannot exploit quantum entanglement between two spacelike separated parties for faster-than-light communication. This can be seen as an immediate consequence of the linearity of quantum mechanics, cf. our discussion below.

Quantum mechanics is linear in two respects: at any given time all measurement probabilities depend linearly on the density matrix of the system; and the state of the system at a given time depends linearly on its initial state. It may seem natural to ask whether quantum mechanics could be an approximation to some underlying nonlinear theory. However, if one tries to modify the theory in this spirit, e.g., by introducing nonlinear time evolution laws for pure states [3], this easily leads to the possibility of superluminal communication [4–6].

This peaceful, but fragile, coexistence between quantum physics and special relativity has led physicists to ask whether the no-signaling condition could be used as an axiom in deriving basic features of quantum mechanics. The answer to this question should at the same time provide insight into what types of modifications of quantum physics are compatible with special relativity.

Here we give a positive answer to the above question. If the usual *static* characteristics of quantum mechanics are assumed, then its *dynamical* rules can be derived from the no-signaling assumption. By static characteristics we mean the following: (i) The states of our systems are described by vectors in a Hilbert space. In particular, this implies the existence of entangled states, which will be essential for our argument. (ii) At any given time we have the usual observables described by projections in the Hilbert space [7], and the probabilities for measurement results are calculated according to the usual trace rule [8]: the probability for getting a positive result for the projector P in the state $|\psi\rangle$ is given by $\langle\psi|P|\psi\rangle = \text{Tr}P|\psi\rangle\langle\psi|$. This implies that if at a given time the system is in states $|\psi_i\rangle$ with probabilities p_i , then the probabilities for measurement results of all the usual observables at this time can be calculated from its density matrix $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$.

We do not make any *a priori* assumption about the time evolution of the system. For example, the states $|\psi\rangle$ could evolve according to some nonlinear wave equation. Then in general the density matrix of a probabilistic mixture of states is not sufficient to determine the dynamics of the system [9], one has to know the individual pure states $|\psi_i\rangle$ and their probabilities p_i .

Let us note that we also do not assume the projection postulate. We will see that the possibility to prepare different probabilistic mixtures of pure states at a distance, which is essential for our argument, already follows from our assumptions (i) and (ii).

Our result is the following: the no-signaling condition together with assumptions (i) and (ii) implies that the dynamics of the theory has to be described by *completely positive* (CP) linear maps [10].

This is equivalent to saying that under the given assumptions quantum physics is essentially the only option since, according to the Kraus representation theorem [10], every CP map can be realized by a quantum-mechanical process,

i.e., by a linear and unitary evolution on a larger Hilbert space. On the other hand, any quantum process corresponds to a CP map. This result is a significant extension of earlier work by one of the authors [5].

Let us first recall how the linearity of standard quantum dynamics prevents the use of entanglement for superluminal communication [11]. Consider two parties, denoted by Alice and Bob, who are spacelike separated, such that all operations performed by Alice commute with all operations performed by Bob. Throughout this paper we will assume that in *this* sense locality is implemented in the algebra of the standard quantum operations, since otherwise the possibility of superluminal signaling would be manifest from the beginning. Can the two parties use a shared entangled state $|\psi_{AB}\rangle$ in order to communicate in spite of their spacelike separation?

The short answer is no, because the situation on Alice's side will always be described by the same reduced density matrix, irrespective of Bob's actions. Alice's reduced density matrix is all that matters as a consequence of the linearity of quantum physics.

Let us discuss this last point in more detail. A question that is frequently raised in the present context is the following: Bob could choose to measure his system in two different bases and thus "project" Alice's system into different pure states depending on the basis he chose and his measurement result. Since it is possible to distinguish two different states in quantum mechanics, at least with some probability, shouldn't it be possible for Alice to infer his choice of basis, at least in some percentage of the cases, which would be dramatic enough?

Of course, the answer is no again. In order to gain information about which basis Bob chose to measure, Alice can only perform some (generalized) measurement on her system. She then has to compare the conditional probabilities for a given result to occur, for the case that Bob measured in the first or in the second basis. But these conditional probabilities will always be exactly the same for both possibilities.

This can be seen in the following way. Suppose that Bob's first choice projects Alice's system into states $|\psi_i\rangle$ with probabilities p_i and his second choice projects it into states $|\phi_k\rangle$ with probabilities q_k . Alice can calculate the probability for her obtained result for every one of the states, and then weight these probabilities with the probability to have this specific state. But because of the linearity of any operation that Alice can perform on her states during her generalized measurement procedure and of the trace rule, her final result will depend only on the density matrix of the probabilistic mixtures, which is the same in both cases, because they were generated from the same entangled state. For an example of how two such mixtures can become distinguishable through a nonlinear (non-quantum-mechanical) evolution, see [4].

Let us note that this argument also implies the nonexistence of a perfect cloner in quantum mechan-

ics because such a machine would allow superluminal communication [12] by making it possible for Alice to discriminate between Bob's choices of basis.

We now show how quantum dynamics can be derived from the no-signaling condition and "quantum statics" as expressed in our assumptions (i) and (ii).

If we consider a subsystem of the whole Universe it will in general be in an entangled state with other parts of the Universe. In particular, a system A may be entangled with another system B which is *spacelike* separated from A , such that their observable algebras commute. The dynamics of the systems has to be such that in spite of this entanglement no superluminal communication between A and B is possible.

Suppose that A and B together are in the entangled state $|\psi\rangle_{AB}$ with reduced density matrix ρ_A for system A . As a consequence of the trace rule, by performing a measurement of his system the observer B also prepares a certain state in A . To see this, remember that the trace rule tells us how to calculate the (joint) probability for measurement results corresponding to any product of projectors $P_A \otimes P_B$, namely, by calculating $\text{Tr}_{AB} \rho_{AB} P_A \otimes P_B$. This also tells us how to calculate the conditional probability to find any P_A , provided that P_B has been found. Namely, we just have to divide the joint probability by the probability to find P_B in the first place. But having a way of calculating the conditional probability for every P_A means that we know the state in A conditional on B having found P_B , since a state can be reconstructed from its expectation values for a linearly independent set of projectors. It is given by $\text{Tr}_B \rho_{AB} P_B / \text{Tr}_{AB} \rho_{AB} P_B$. Note that to arrive at this conclusion we did not have to make use of the usual projection postulate.

Actually, *every* probabilistic mixture of pure states corresponding to the density matrix ρ_A can be prepared via appropriate measurements on B [5,13]. We will give a proof of this statement in the last part of this Letter.

Consider two such probabilistic mixtures $\{P_{\psi_i}, p_i\}$ and $\{P_{\phi_j}, q_j\}$, where P_{ψ_k} is the projector corresponding to the pure state $|\psi_k\rangle$ and p_k is its probability, such that

$$\sum_i p_i P_{\psi_i} = \sum_j q_j P_{\phi_j} = \rho_A. \quad (1)$$

According to the no-signaling principle there should be no way for the observer in A to distinguish these different probabilistic mixtures.

A general dynamical evolution in system A is of the form $g: P_{\psi} \rightarrow g(P_{\psi})$, where, most importantly, g is not necessarily linear. Furthermore, $g(P_{\psi})$ does not have to be a pure state, since system A could become entangled with its environment, or $|\psi\rangle$ could evolve into a probabilistic mixture of pure states. As mentioned above, even if system A is entangled with its environment, the trace rule implies that at any given moment the results of measurements on A will be completely determined by the reduced density

matrix of the system. In this case we define $g(P_\psi)$ to be the reduced density matrix of A . If $|\psi\rangle$ evolves into a probabilistic mixture, we define $g(P_\psi)$ to denote the corresponding density matrix.

Under such dynamics the probabilistic mixture $\{P_{\psi_k}, p_k\}$ goes into another probabilistic mixture $\{g(P_{\psi_k}), p_k\}$. Therefore the two final density matrices after the action of g on two different probabilistic mixtures $\{P_{\psi_i}, p_i\}$ and $\{P_{\phi_j}, q_j\}$ are

$$\begin{aligned}\rho'_A(\{P_{\psi_i}, p_i\}) &= \sum_i p_i g(P_{\psi_i}), \\ \rho'_A(\{P_{\phi_j}, q_j\}) &= \sum_j q_j g(P_{\phi_j}),\end{aligned}\quad (2)$$

which *a priori* can be different.

Let us note that of course ρ'_A is always linear in the $g(P_{\psi_i})$, because of the definition of the density matrix corresponding to a probabilistic mixture. However g , which is *a priori* defined only on pure states, could have an arbitrary functional dependence on the P_{ψ_i} . We will now show that this dependence is constrained to be linear by the no-signaling condition.

Let us recall that according to our assumptions the results of all standard quantum measurements in A at a given time are determined by the reduced density matrix ρ'_A . This means that the density matrix ρ'_A at any later time has to be the same for all probabilistic mixtures corresponding to a given initial density matrix ρ_A , since otherwise Alice could distinguish different mixtures at least with some probability. That is, ρ'_A has to be a *function of ρ_A only*.

We can therefore write

$$\rho'_A = g(\rho_A) = g\left(\sum_i p_i P_{\psi_i}\right), \quad (3)$$

i.e., g now extends to density matrices. Equations (2) and (3) together imply the linearity of g :

$$g\left(\sum_i p_i P_{\psi_i}\right) = \sum_i p_i g(P_{\psi_i}). \quad (4)$$

Positivity of g is necessary in order to ensure that $g(\rho_A)$ is again a valid density matrix, i.e., to ensure the positivity of all probabilities calculated from it.

As we have made no specific assumptions about the system A apart from the fact that it can be entangled with some other spacelike separated system, this means that the dynamics of our theory has to be described by linear maps on density matrices in general.

Linearity and *positivity* already imply *complete positivity* [10] in the present framework. To see this, consider again two arbitrary subsystems A and B which may again be in an entangled state $|\psi\rangle_{AB}$. It is conceivable that system A is changed locally (i.e., the system evolves, is measured, etc.), which is described by some linear and positive operation g_A , while *nothing* happens in B . This formally corresponds to the operation $g_A \otimes \mathbb{1}_B$ on the whole system. The joint operation $g_A \otimes \mathbb{1}_B$ should take the density matrix of the composite system ρ_{AB} into another valid (i.e., positive)

density matrix, whatever the dimension of the system B . But this is exactly the definition of *complete positivity* for the map g_A [10]. If g_A is positive but not CP, then by definition there is always some entangled state ρ_{AB} for which $g_A \otimes \mathbb{1}_B$ applied to ρ_{AB} is no longer a positive density matrix and thus leads to nonphysical results such as negative probabilities.

In this way the existence of entangled states and the requirements of positivity and linearity force us to admit only completely positive dynamics. As mentioned above, this means that under the given assumptions quantum dynamics is essentially the *only* option since any CP map can be realized by a quantum-mechanical process, while, on the other hand, any quantum-mechanical process corresponds to a CP map [14].

There are three crucial ingredients in our argument: the existence of entanglement, the trace rule, and the no-signaling condition. Specifically, the trace rule leads to the preparation at a distance of probabilistic mixtures and thus, as it were, to the right-hand side of Eq. (4). On the other hand, the no-signaling condition tells us that the dynamics is allowed to depend only on the reduced density matrix, which leads to the left-hand side of Eq. (4). In the derivation of complete positivity, we have also used the assumption that the identity operation on a subsystem is a permitted dynamical evolution.

Nonlinear modifications of quantum mechanics [15,16] have to give up at least one of these assumptions. For instance, if the dynamics is allowed to depend on the reduced density matrix ρ_A in a nonlinear way, then it is clear that ρ_A cannot correspond to a probabilistic mixture of pure states, cf. [15]. But ρ_A will correspond to such a mixture whenever the observer in B chooses to make appropriate measurements, as long as we believe in the trace rule, according to our above argument. This seems to imply that, at least for separated systems, the trace rule has to be modified in such a nonlinear theory.

Another example would be a theory where some entangled states are *a priori* excluded. In this case some non-CP maps might be permissible. An extreme example would be a theory without entanglement. Such a theory would of course be in conflict with experiments. An example of a linear, positive, but non-CP map consistent with no-signaling is the transposition of the density matrix of the whole Universe (physically corresponding to a time reversal). However, in this case the identity operation on a subsystem is not an allowed dynamics.

In this Letter we have taken quantum statics as a starting point. Svetlichny [17] recently investigated the possibility of deriving the Hilbert space structure itself in the context of quantum logic, using Lorentz covariance as an ingredient.

For completeness, let us finally show that any mixture corresponding to a given density matrix can be prepared at a distance from any entangled state with the correct reduced density matrix [5,13]. Let us denote the system

under consideration by A and the remote system by B . Let us denote the eigenvector representation of ρ_A by $\sum_{k=1}^r \lambda_k |v_k\rangle\langle v_k|$. Since the joint state $|\psi\rangle_{AB}$ needs to have the correct reduced density matrix, it must have a Schmidt decomposition $|\psi\rangle_{AB} = \sum_{k=1}^r \sqrt{\lambda_k} |v_k\rangle |g_k\rangle$, where the $|g_k\rangle$ are orthonormal states of system B .

We want to show that *any* decomposition of ρ_A as a mixture of pure states can be prepared from this state by operations on system B only. To this end, consider an arbitrary decomposition $\rho_A = \sum_{i=1}^m x_i |\psi_i\rangle\langle\psi_i|$, where in general $m > r$. Clearly this decomposition could be obtained from a state $|\phi\rangle_{AB} = \sum_{i=1}^m \sqrt{x_i} |\psi_i\rangle |\alpha_i\rangle$, with the $|\alpha_i\rangle$ being an orthonormal basis of a m -dimensional Hilbert space H_m .

It seems that we now require a larger Hilbert space in location B in order to accommodate all the orthonormal $|\alpha_i\rangle$. But the state $|\phi\rangle_{AB}$ must also have a Schmidt representation $|\phi\rangle_{AB} = \sum_{k=1}^r \sqrt{\lambda_k} |v_k\rangle |h_k\rangle$, with $|h_k\rangle$ being orthonormal states in B . This implies that $|\phi\rangle_{AB}$ and $|\psi\rangle_{AB}$ are connected by a unitary transformation on B alone: $|\phi\rangle_{AB} = \mathbb{1}_A \otimes U_B |\psi\rangle_{AB}$. The dimension of the support of the reduced density matrix ρ_B is the same for both states, since it is given by the dimension of the support of ρ_A .

Thus one can prepare $|\phi\rangle_{AB}$ from any state with the correct reduced density matrix by extending the system B locally to m dimensions (using an appropriate ancilla), and then perform the required measurement in the basis of the $|\alpha_i\rangle$. This will correspond to a generalized measurement [8] on the original r -dimensional system. In this way every possible decomposition of ρ_A can be prepared at a distance.

In conclusion, we have shown that the basic dynamical rule of quantum physics can be derived from its static properties and the condition of no superluminal communication. This result puts significant constraints on nonlinear modifications of quantum physics. It is clearly difficult to modify just parts of the whole structure. More universal departures from the formalism may still be possible without violating the no-signaling condition. We would like to mention the related recent work by Mielnik [18].

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