

Programmable quantum-state discriminators with simple programs

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We describe a class of programmable devices that can discriminate between two quantum states. We consider two cases. In the first, both states are *unknown*. One copy of each of the unknown states is provided as an input, or program, for the two program registers, and the data state, which is guaranteed to be prepared in one of the program states, is fed into the data register of the device. This device will then tell us, in an optimal way, which of the templates stored in the program registers the data state matches. In the second case, we know one of the states while the other is *unknown*. One copy of the unknown state is fed into the single program register, and the data state which is guaranteed to be prepared in either the program state or the known state, is fed into the data register. The device will then tell us, again optimally, whether the data state matches the template or is the known state. We determine two types of optimal devices. The first performs discrimination with minimum error, and the second performs optimum unambiguous discrimination. In all cases we first treat the simpler problem of only one copy of the data state and then generalize the treatment to n copies. In comparison to other works we find that providing $n > 1$ copies of the data state yields higher success probabilities than providing $n > 1$ copies of the program states.

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I. INTRODUCTION

Quantum state discrimination [1] is a basic tool for many tasks in quantum information and quantum communication. In the prototype problem a quantum processor generates a quantum system as its output which is in one of a set of known states but we do not know which and want to determine the actual state. If the possible states are not orthogonal this cannot be done with 100% probability of success since the cloning of quantum states is impossible. There are two basic strategies to accomplish state discrimination. In the first, every time a measurement is performed we want to identify the state of the output with one of the possible states. Clearly, errors must be permitted and in the error minimizing strategy the optimum measurement is such that the probability of error is minimum. The case of discriminating with minimum error between two possible states was treated in the pioneering work by Helstrom [2]. More recently, the interest was focused on the unambiguous discrimination. In this strategy we are not permitted to make an erroneous identification of the state. The cost associated with this condition is that sometimes we fail to identify the state altogether. In the optimum strategy the probability of failure is a minimum. The optimal value of the failure probability for two known and equally likely pure states was obtained by Ivanovic, Dieks, and Peres (IDP bound [3–5]). Later Jaeger and Shimony [6] generalized the IDP bound for arbitrary preparation probabilities of the states, i.e., for arbitrary prior probabilities of the two possible states.

The actual state-distinguishing device for two *known* states depends on the two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, i.e., these two states are “hard wired” into the machine. Another approach is

to supply the information about the states to be distinguished as inputs, in particular as quantum inputs. That is, one encodes the information about the states one wants to distinguish into a quantum state, which is then a kind of quantum program, that is sent into the discriminator at the same time as the particle whose state is to be identified. The first such device was proposed by Dušek and Bužek [7]. This device distinguishes the two states $\cos(\phi/2)|0\rangle \pm \sin(\phi/2)|1\rangle$, and the angle ϕ is encoded into a one-qubit program state in a somewhat complicated way. The device does not achieve the maximum possible success probability for all input states, but its success probability, averaged over the angle ϕ , is greater than 90% of the optimal value.

In a series of recent works, Fiurášek *et al.* investigated a closely related programmable device that can perform a von Neumann projective measurement in any basis, the basis being specified by the program. Both deterministic and probabilistic approaches were explored [8], and experimental versions of both the state discriminator and the projective measurement device were realized [9]. Sasaki *et al.* developed a related device, which they called a quantum matching machine [10]. Its input consists of K copies of two equatorial qubit states, which are called templates, and N copies of another equatorial qubit state $|f\rangle$. The device determines which of the two template states $|f\rangle$ is closest. This device does not employ the unambiguous discrimination strategy, but optimizes an average score that is related to the fidelity of the template states and $|f\rangle$. Programmable quantum devices to accomplish other tasks have been explored by a number of authors [11–22].

Recently two of us proposed an approach to a programmable state discriminating machine in which the program is

related in a simple way to the states $|\psi_1\rangle$ and $|\psi_2\rangle$ that one is trying to distinguish [18]. In order to introduce the range of problems to be dealt with here and to establish a uniform terminology we begin with a brief review of the problem investigated in that work. A system of three qubits, labeled A , B , and C , is considered and we assume that the qubit A is prepared in the state $|\psi_1\rangle$, and the qubit C is prepared in the state $|\psi_2\rangle$. Qubit B is guaranteed to be prepared in either $|\psi_1\rangle$ or $|\psi_2\rangle$, with a probability η_1 of being in $|\psi_1\rangle$ and a probability $\eta_2=1-\eta_1$ of being in $|\psi_2\rangle$. The states $|\psi_1\rangle$ or $|\psi_2\rangle$ are different and unknown. Our task is to find whether the state of qubit B is $|\psi_1\rangle$ or $|\psi_2\rangle$. One way of looking at this problem is to view the qubits A and C as a program, and we refer to their states as program states. They are sent into the program registers, also denoted by A and C , of some device, called a programmable state discriminator, and the third, unknown qubit is sent into the data register, denoted by B , of this device. The state of qubit B is called the data state. The device then identifies the data state with one of the two unknown program states with optimal probability of success. Despite the complete lack of classical information about the program states, matching the data state to one or the other program state is still possible due to symmetry properties that are intrinsically quantum mechanical, and are similar to those first employed by Barnett *et al.* [23] for the purpose of comparing unknown states. The program is the most elementary possible, and consists of copies of the states one is trying to distinguish. It should be noted at this point that our terminology is not unique. In various contexts the program states are also called reference states or templates, and the data state the sample. The programmable discrimination can then also be called template matching or state identification. However, in what follows we will use our original terminology introduced in Ref. [18], i.e., we will use the terms, program state and data state throughout the paper.

The original results in Ref. [18] were for unambiguous discrimination and for qubit data and program states. They have recently been extended to qudits by Hayashi *et al.*, both for optimum unambiguous discrimination [19] and for minimum error discrimination [20]. Their investigations are restricted to equal prior probabilities, where the data state equally likely matches one or the other of the program states, but they also dealt with the case in which an arbitrary number of copies is provided for each of the two program states.

In the present paper we generalize the programmable state discriminator introduced in Ref. [18] in several other directions and develop a comparative study of programmable state discriminators based on the two measurement strategies of minimum-error discrimination and optimum unambiguous discrimination. For this purpose in Sec. II we first reformulate the problem of programmable discrimination of pure states as a problem of discrimination between two mixed quantum states. Section III is devoted to the case that both of the pure states to be discriminated are unknown so we need a program state for each. In Sec. III A we treat the error minimizing version of the programmable state discriminator, considering both a joint measurement on all three qubits, and also a measurement prescription that is restricted to two-qubit measurements only. In Sec. III B we rederive the results of Ref. [18] for the unambiguous version of the pro-

grammable state discriminator partly for comparison's sake but also using the consistent approach based on the equivalent mixed state discrimination problem. It should be noted, in this context, that the results of Ref. [18] were obtained in a somewhat ad hoc manner and the current approach gives a solid foundation to those results. We also compare the optimal probabilities obtained in Secs. III A and III B for the two possible strategies. In Sec. IV we fill another gap and show how to construct devices that can optimally discriminate between one known and one unknown state using both minimum error and optimum unambiguous strategies. That is, we know what $|\psi_1\rangle$ is, but do not know $|\psi_2\rangle$. Then we need a reference state only for the unknown state, which constitutes the program in this case. We can say that this line of investigation characterizes the quality of the source that produces the states to be discriminated, or the quality of our knowledge about the source, respectively. If both possible states are known (the original IDP and Helstrom problem) there is no need for a program, the states are hard wired into the optimal device. If one of the states is known we need a program for the unknown state while the other is hard wired into the device and if both states are unknown we need a program for both.

In Secs. V and VI, we investigate the effect of the available resources on the performance of these devices. Suppose that instead of a single copy of the data state we are provided with n copies. In Sec. V we generalize the two unknown qubit scenarios of Sec. III for the case when n copies of the data state, and one copy each of the program states, are provided. In Sec. VI we provide a similar generalization of the one known and one unknown qubit scenarios treated in Sec. IV. Finally, in Sec. VII we conclude with a brief discussion of how these results can be used to characterize the preparation quality (source quality) and to quantify the available resources.

II. DISCRIMINATION OF UNKNOWN PURE STATES AND ITS CONNECTION TO THE DISCRIMINATION OF MIXED STATES

Let us begin by briefly reviewing the problem that was originally addressed in Ref. [18]. We consider a system of three qubits, labeled A , B , and C , and assume that the qubit A is prepared in the state $|\psi_1\rangle$, and the qubit C is prepared in the state $|\psi_2\rangle$. Qubit B is guaranteed to be prepared in either $|\psi_1\rangle$ or $|\psi_2\rangle$, with a probability η_1 of being in $|\psi_1\rangle$ and a probability $\eta_2=1-\eta_1$ of being in $|\psi_2\rangle$. The states $|\psi_1\rangle$ or $|\psi_2\rangle$ are different and unknown. Our task is to find whether the state of qubit B is $|\psi_1\rangle$ or $|\psi_2\rangle$. One way of looking at this problem is to view the qubits A and C as a program. They are sent into the program register of some device, called a programmable state discriminator, and the third, unknown qubit is sent into the data register of this device. The device then tells us, with an optimal probability of success, which one of the two program states the unknown state of the qubit in the data register corresponds to. We can consider this problem as a task in measurement optimization. We want to find an optimal measurement strategy that, with a maximum probability of success, tells us which one of the two program states,

stored in the program register, matches the unknown state, stored in the data register. In Ref. [18] only unambiguous discrimination was treated, in which the measurement is allowed to return an inconclusive result but never an erroneous one. Here we want to investigate the measurement strategy of minimum-error discrimination, as well. In general, we want to determine the best possible measurement for identifying the state of the qubit B . Our task is then reduced to the following measurement optimization problem. One has two input states

$$\begin{aligned} |\Psi_1\rangle &= |\psi_1\rangle_A |\psi_1\rangle_B |\psi_2\rangle_C, \\ |\Psi_2\rangle &= |\psi_1\rangle_A |\psi_2\rangle_B |\psi_2\rangle_C, \end{aligned} \quad (2.1)$$

where the subscripts A and C refer to the program registers (A contains $|\psi_1\rangle$ and C contains $|\psi_2\rangle$), and the subscript B refers to the data register. Our goal is to optimally distinguish between these inputs, with respect to some reasonable criteria, keeping in mind that one has no knowledge of $|\psi_1\rangle$ and $|\psi_2\rangle$ beyond their *a priori* probabilities.

Assuming the states $|\psi_1\rangle$ and $|\psi_2\rangle$ to be completely unknown, we have to find the measurement strategy that is optimal on average. Thus, we have to take the average of the input with respect to all possible qubit states. The problem is then equivalent to distinguishing between two mixed states, given by the density operators,

$$\rho_1 = \{|\Psi_1\rangle\langle\Psi_1|\}_{\text{av}}, \quad (2.2)$$

$$\rho_2 = \{|\Psi_2\rangle\langle\Psi_2|\}_{\text{av}}, \quad (2.3)$$

that occur with the prior probabilities η_1 and η_2 , respectively. Any state of a particular qubit (A , B , or C) can be represented using the Bloch parametrization given by $|\psi_i\rangle = \cos(\theta_i/2)|0\rangle + e^{i\phi_i} \sin(\theta_i/2)|1\rangle$ ($i=1,2$), with $|0\rangle$ and $|1\rangle$ denoting an arbitrary set of orthonormal basis states. Here θ and ϕ are the polar and azimuthal angle on the Bloch sphere. After performing the averaging with respect to all possible values of θ and ϕ we arrive at

$$\rho_1 = \frac{1}{6} P_{AB}^{\text{sym}} \otimes I_C, \quad (2.4)$$

$$\rho_2 = \frac{1}{6} I_A \otimes P_{BC}^{\text{sym}}, \quad (2.5)$$

where $P_{AB}^{\text{sym}} = \sum_{i=1}^3 |u_i\rangle_{AC} \langle u_i|$ and $P_{BC}^{\text{sym}} = \sum_{i=1}^3 |\bar{u}_i\rangle_{BC} \langle \bar{u}_i|$ are the projectors onto the symmetric subspaces of the corresponding qubits, AB and BC , respectively. Here we used the two-qubit basis states

$$|u_1\rangle_{AB} = |0\rangle_A |0\rangle_B, \quad |u_2\rangle_{AB} = \frac{|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B}{\sqrt{2}}, \quad (2.6)$$

$$|u_3\rangle_{AB} = |1\rangle_A |1\rangle_B, \quad |\bar{u}_2\rangle_{AB} = \frac{|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B}{\sqrt{2}}, \quad (2.7)$$

and the analogous expressions for the qubit combination BC . Due to the symmetry of the state ρ_1 with respect to interchanging qubits A and B the antisymmetric state $|u_4\rangle_{AB}$ does

not enter the expression for the density operator. Equations (2.4) and (2.5) reduce our state identification problem to the problem of discriminating between these two mixed states.

When one of the two states that we want to distinguish is known, we arrive at a simpler variant of the discrimination problem. We do not need to provide a template for the known state, and one of the program registers, say A , can be eliminated from the problem. It is convenient to define the single-qubit basis states in such a way that the known pure state serves as one of the basis states, denoted by $|0\rangle$, so $|\psi_1\rangle = |0\rangle$. We then have to distinguish two cases, the qubit B is either in the state $|0\rangle_B$, occurring with the prior probability η_1 , or it is in the unknown state of the qubit C , occurring with the prior probability η_2 . These two cases correspond to the density operators

$$\begin{aligned} \rho'_1 &= |0\rangle_{BB} \langle 0| \otimes \{|\psi\rangle_{CC} \langle \psi|\}_{\text{av}} = \frac{1}{2} |0\rangle_{BB} \langle 0| \otimes I_C \\ &= \frac{1}{2} \{ |u_1\rangle_{BC} \langle u_1| + |v_2\rangle_{BC} \langle v_2| \}, \end{aligned} \quad (2.8)$$

$$\rho'_2 = \{ |\psi\rangle_B \langle \psi| \}_{CB} \{ |\psi\rangle_C \langle \psi| \}_{\text{av}} = \frac{1}{3} P_{BC}^{\text{sym}}. \quad (2.9)$$

Here

$$|v_2\rangle_{AB} = |0\rangle_A |1\rangle_B. \quad (2.10)$$

In addition, we introduce

$$|\bar{v}_2\rangle_{AB} = |1\rangle_A |0\rangle_B, \quad (2.11)$$

together with the analogous expression for the qubit combination BC . $\{|u_2\rangle, |\bar{u}_2\rangle\}$ and $\{|v_2\rangle, |\bar{v}_2\rangle\}$ form alternative bases for the subspace with exactly one qubit in the state $|1\rangle$. They will prove useful later when we consider the various discrimination scenarios in the following sections.

After these preliminary considerations we are now in a position to investigate different possible measurements for identifying the state of the qubit B , i.e., for distinguishing between the two density operators given by Eqs. (2.4) and (2.5) or, alternatively, by Eqs. (2.8) and (2.9). Before doing so, we briefly recall the underlying theoretical concepts for treating the strategies of discriminating two mixed states. Any measurement suitable for distinguishing between the mixed states ρ_1 and ρ_2 , occurring with the prior probabilities η_1 and $\eta_2 = 1 - \eta_1$, respectively, can be formally described with the help of three positive detection operators Π_0 , Π_1 , and Π_2 , whose sum is the identity,

$$\Pi_0 + \Pi_1 + \Pi_2 = I. \quad (2.12)$$

These operators are defined in such a way that for $j = 1, 2$ $\text{Tr}(\rho \Pi_j)$ is the probability to infer from the measurement that the system is in the state ρ_j if it has been prepared in a state ρ , while $\text{Tr}(\rho \Pi_0)$ is the probability that the measurement result is inconclusive, i.e., that the measurement fails to give a definite answer. When all detection operators are projectors, the measurement is a von Neumann measurement, otherwise it is a generalized measurement based on a positive operator-valued measure (POVM). Once the detec-

tion operators of a generalized measurement have been found, Neumark's theorem guarantees that schemes for actually realizing the measurement can be devised by determining suitable projections in an enlarged Hilbert space that results from appending an ancilla to the original system [24,25].

The above POVM is appropriate for unambiguous state discrimination. For *minimum-error discrimination* inconclusive results do not occur, so that

$$\Pi_0 = 0, \quad (2.13)$$

and we require that the probability of errors in the discrimination procedure is a minimum. For two mixed states this problem was originally solved by Helstrom [2]. The error probability is always larger than zero unless the states to be distinguished are orthogonal, and it can be expressed as

$$\begin{aligned} P_{\text{err}} &= \eta_1 \text{Tr}(\rho_1 \Pi_2) + \eta_2 \text{Tr}(\rho_2 \Pi_1) \\ &= \eta_1 + \text{Tr}[(\eta_2 \rho_2 - \eta_1 \rho_1) \Pi_1], \end{aligned} \quad (2.14)$$

where in the second line Eqs. (2.12) and (2.13) have been used, as well as the relation $\eta_2 = 1 - \eta_1$. After introducing the operator

$$\Lambda = \eta_2 \rho_2 - \eta_1 \rho_1 = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k| \quad (2.15)$$

it is obvious that the minimum of the error probability is obtained when Π_1 is the projector onto those eigenstates $|\phi_k\rangle$ of Λ that belong to negative eigenvalues λ_k . The optimum detection operators therefore read

$$\Pi_1^{\text{opt}} = \sum_{k < k_0} |\phi_k\rangle\langle\phi_k|, \quad \Pi_2^{\text{opt}} = \sum_{k \geq k_0} |\phi_k\rangle\langle\phi_k|, \quad (2.16)$$

where $\lambda_k < 0$ for $1 \leq k < k_0$ and $\lambda_k \geq 0$ for $k \geq k_0$. Clearly, these two operators are projections, and the optimal minimum-error measurement for discriminating between two quantum states is, therefore, always a von Neumann measurement. The resulting minimum error probability $P_{\text{err}}^{\text{min}} = P_E$ is given in Ref. [2] by

$$P_E = \frac{1}{2} (1 - \text{Tr}|\eta_2 \rho_2 - \eta_1 \rho_1|) = \frac{1}{2} \left(1 - \sum_k |\lambda_k| \right). \quad (2.17)$$

In *optimum unambiguous discrimination* which is the other frequently used strategy, errors are not allowed to occur. This requirement is equivalent to

$$\rho_1 \Pi_2 = \rho_2 \Pi_1 = 0 \quad (2.18)$$

(see, for example [1]). In the optimum measurement scheme the failure probability, i.e., the probability for getting an inconclusive outcome, is minimized, taking into account the constraint that the eigenvalues of the operator $\Pi_0 = I - \Pi_1 - \Pi_2$ are non-negative. The failure probability is always non-zero unless the states to be discriminated are orthogonal. It can be expressed as

$$\begin{aligned} Q_{\text{fail}} &= \eta_1 \text{Tr}(\rho_1 \Pi_0) + \eta_2 \text{Tr}(\rho_2 \Pi_0) = 1 - \eta_1 \text{Tr}(\rho_1 \Pi_1) \\ &\quad - \eta_2 \text{Tr}(\rho_2 \Pi_2) = 1 - P_{\text{succ}}, \end{aligned} \quad (2.19)$$

where Eqs. (2.12) and (2.18) have been used, and where we

also introduced the success probability P_{succ} of the measurement. Optimum unambiguous discrimination between two mixed states is an issue of ongoing theoretical research [26–32]. In contrast to minimum-error discrimination, there does not exist a compact formula expressing the minimum probability of inconclusive results, i.e., the minimum failure probability, for unambiguously discriminating two mixed states that are completely arbitrary. However, analytical solutions can be obtained for certain special classes of density operators, including the cases that are of interest for this paper.

III. TWO-QUBIT PROGRAM, SINGLE COPY OF THE DATA STATE

A. Minimum-error discrimination strategy

1. Joint measurement on all three qubits

We begin by investigating the measurement that discriminates, with minimum probability of error, between the density operators given by Eqs. (2.4) and (2.5). For this purpose we define the orthonormal basis states in the eight-dimensional Hilbert space spanned by the three qubits as

$$\begin{aligned} |1\rangle &= |0\rangle_A |0\rangle_B |0\rangle_C, & |2\rangle &= |0\rangle_A |0\rangle_B |1\rangle_C, \\ |3\rangle &= |0\rangle_A |1\rangle_B |0\rangle_C, & |4\rangle &= |1\rangle_A |0\rangle_B |0\rangle_C, \\ |5\rangle &= |0\rangle_A |1\rangle_B |1\rangle_C, & |6\rangle &= |1\rangle_A |0\rangle_B |1\rangle_C, \\ |7\rangle &= |1\rangle_A |1\rangle_B |0\rangle_C, & |8\rangle &= |1\rangle_A |1\rangle_B |1\rangle_C. \end{aligned} \quad (3.1)$$

The numbering of the states is essentially the binary number formed by the bit values on the right-hand side shifted by 1. Note, however, that in the case of $|4\rangle$ and $|5\rangle$ the order is reversed. Expanding the expressions for ρ_1 and ρ_2 in this basis and introducing the notations,

$$|r_1\rangle = \frac{|3\rangle + |4\rangle}{\sqrt{2}}, \quad |r_2\rangle = \frac{|5\rangle + |6\rangle}{\sqrt{2}}, \quad (3.2)$$

$$|s_1\rangle = \frac{|2\rangle + |3\rangle}{\sqrt{2}}, \quad |s_2\rangle = \frac{|6\rangle + |7\rangle}{\sqrt{2}}, \quad (3.3)$$

we obtain the spectral representations

$$\rho_1 = \frac{1}{6} \left(\sum_{l=1}^2 |r_l\rangle\langle r_l| + |1\rangle\langle 1| + |2\rangle\langle 2| + |7\rangle\langle 7| + |8\rangle\langle 8| \right), \quad (3.4)$$

$$\rho_2 = \frac{1}{6} \left(\sum_{l=1}^2 |s_l\rangle\langle s_l| + |1\rangle\langle 1| + |4\rangle\langle 4| + |5\rangle\langle 5| + |8\rangle\langle 8| \right). \quad (3.5)$$

When we express the operator $\Lambda = \eta_2 \rho_2 - \eta_1 \rho_1$ with the help of the basis states given by Eq. (3.1), we arrive at an eight-dimensional square matrix which is block-diagonal if the columns and rows are numbered according to the numbering of the basis states. It can be written as

$$\Lambda = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_3 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_1 \end{pmatrix}, \quad (3.6)$$

where $L_1 = (\eta_2 - \eta_1)/6$ and

$$L_3 = \frac{1}{12} \begin{pmatrix} \eta_2 - 2\eta_1 & \eta_2 & 0 \\ \eta_2 & \eta_2 - \eta_1 & -\eta_1 \\ 0 & -\eta_1 & 2\eta_2 - \eta_1 \end{pmatrix}. \quad (3.7)$$

Since the eigenvalues of L_3 are $(\eta_2 - \eta_1)/6$ and

$$\lambda_{\pm} = \frac{1}{12} [\eta_2 - \eta_1 \pm \sqrt{(\eta_2 - \eta_1)^2 + 3\eta_1\eta_2}], \quad (3.8)$$

the complete set of eigenvalues of the operator Λ is given by

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda_-, \quad \lambda_3 = \lambda_4 = \lambda_+, \\ \lambda_k = \frac{\eta_2 - \eta_1}{6} \quad (5 \leq k \leq 8). \end{aligned} \quad (3.9)$$

The corresponding eigenstates, $|\phi_k\rangle$, are found to be $|\phi_7\rangle = |1\rangle$, $|\phi_8\rangle = |8\rangle$,

$$\begin{aligned} |\phi_1\rangle &= \frac{a^-|2\rangle + |3\rangle + b^-|4\rangle}{\sqrt{1 + (a^-)^2 + (b^-)^2}}, \\ |\phi_3\rangle &= \frac{a^+|2\rangle + |3\rangle + b^+|4\rangle}{\sqrt{1 + (a^+)^2 + (b^+)^2}}, \\ |\phi_5\rangle &= \frac{|2\rangle + |3\rangle + |4\rangle}{\sqrt{3}}, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} a^{\pm} &= \pm \sqrt{(\eta_1 - \eta_2)^2 + 3\eta_1\eta_2} - \eta_1, \\ b^{\pm} &= \mp \sqrt{(\eta_1 - \eta_2)^2 + 3\eta_1\eta_2} - \eta_2. \end{aligned} \quad (3.11)$$

The eigenstates $|\phi_2\rangle$, $|\phi_4\rangle$, $|\phi_6\rangle$ follow from replacing the ordered set $\{|2\rangle, |3\rangle, |4\rangle\}$ by the ordered set $\{|5\rangle, |6\rangle, |7\rangle\}$ in the expressions given by Eq. (3.10). By inserting the eigenvalues of Λ into Eq. (2.17) we find after a little algebra that the minimum probability of error can be written in the following compact way,

$$P_E = \eta_{\min} \left(1 - \frac{1}{2} \frac{\eta_{\max}}{\eta_{\max} - \eta_{\min} + \sqrt{1 - \eta_{\max}\eta_{\min}}} \right), \quad (3.12)$$

where η_{\max} (η_{\min}) is the larger (smaller) of η_1 and η_2 . This result is in agreement with the one derived in Ref. [20] for the special case that $\eta_{\max} = \eta_{\min} = 1/2$.

The above expression lends itself to a transparent interpretation. The error probability P_e would be η_{\min} if we did not perform any measurement at all but would simply guess, always choosing the state whose *a priori* probability is larger. The factor in the bracket, multiplying η_{\min} , is the im-

provement due to the optimized measurement. It is a slowly varying function of the prior probabilities, its value lying between 0.71 and 0.75 in the entire $0 < \eta_{\min} < 1/2$ interval. To be specific, let us assume that the qubits are labeled in such a way that η_1 is the smaller of the two prior probabilities, i.e., that $\eta_1 \leq 0.5$. In this case λ_1 and λ_2 are the only negative eigenvalues, and the optimum detection operators Π_1^{opt} and Π_2^{opt} for minimum-error identification take the form given by Eq. (2.16) with $k_0 = 3$. This means that the qubit B is inferred to be in the state of qubit A when a projection onto the subspace spanned by the eigenstates of Π_1^{opt} is successful, and after successful projection onto the complementary subspace it is inferred to be in the state of qubit C .

From the structure of the eigenstates $|\phi_k\rangle$ determining the optimum detection operators, and from the definition of the basis states, given by Eq. (3.1), it is obvious that the smallest possible error probability, P_E , can only be obtained by performing a joint measurement on all three qubits simultaneously. The question therefore naturally arises as to what is the smallest value of the error probability achievable under the restriction that only joint measurements on two qubits are allowed. In the following we study this problem. This situation is worth examining for two reasons. First, two-qubit measurements are easier to perform than three-qubit ones. Second, by comparing the results of the two- and three-qubit measurements, we see how the additional quantum information contained in the third qubit affects the result.

2. Restriction to two-qubit-measurements

First let us assume that the qubit C is not accessible, but that we are able to perform a joint measurement on the qubits A and B . This would be the case, for example, if a copy of only one of the two states we are trying to distinguish is provided. Starting again from Eqs. (2.4) and (2.5), the problem of identifying the state of B is then equivalent to discriminating between the two reduced density operators,

$$\text{Tr}_C \rho_1 = \frac{1}{3} P_{AB}^{\text{sym}}, \quad (3.13)$$

$$\text{Tr}_C \rho_2 = \frac{1}{4} J_A \otimes I_B, \quad (3.14)$$

with the prior probabilities η_1 and η_2 , respectively.

The operator $\tilde{\Lambda} = \eta_2 \text{Tr}_C \rho_2 - \eta_1 \text{Tr}_C \rho_1$, relevant for minimum-error discrimination, is now given by

$$\tilde{\Lambda} = \left(\frac{\eta_2}{4} - \frac{\eta_1}{3} \right) \sum_{i=1}^3 |u_i\rangle_{AB} \langle u_i| + \frac{\eta_2}{4} |\bar{u}_2\rangle_{AB} \langle \bar{u}_2|. \quad (3.15)$$

When $\eta_1 \leq 3\eta_2/4$ (or $\eta_1 \leq 3/7$) all four eigenvalues of $\tilde{\Lambda}$ are positive, and from Eq. (2.16) the optimum detection operators are obtained as $\Pi_2^{\text{opt}} = I$ and $\Pi_1^{\text{opt}} = 0$. Hence the minimum error probability is achieved by guessing that the quantum system is always in the state that is more probable, in this case $|\psi_2\rangle$, without performing any measurement at all. This is a special situation, described earlier [33], which has been

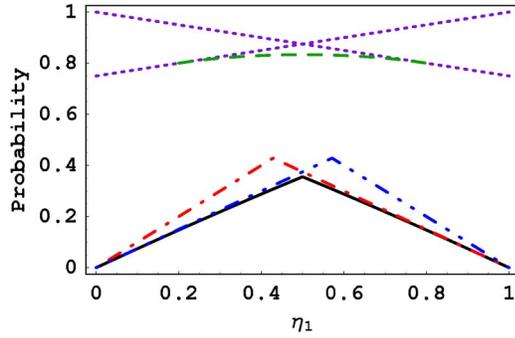


FIG. 1. (Color online) Comparison of the optimum performances of the error minimizing strategy and the unambiguous discrimination strategy for the discrimination of two unknown pure states. The minimum error probabilities P_E resulting from a three-qubit measurement (full line) and P_E^{AB} , P_E^{BC} resulting from two-qubit measurements (dashed-dotted and dashed-double dotted line, respectively) are compared to the failure probabilities for unambiguous discrimination Q_F^{AB} , Q_F^{BC} (dotted lines) and Q_F^{POVM} (dashed line). The error and failure probabilities are plotted vs the prior probability η_1 that the state of the data qubit B matches the state of the program qubit A .

observed in connection with a different problem of two-qubit-discrimination [34].

On the other hand, for $\eta_1 \geq 3\eta_2/4$ (or $\eta_1 \geq 3/7$) we readily find that $\Pi_2^{\text{opt}} = |\bar{u}_2\rangle_{AB} \langle \bar{u}_2|$ and $\Pi_1^{\text{opt}} = \sum_{i=1}^3 |u_i\rangle_{AB} \langle u_i|$, i.e., that the error probability is smallest when the qubit B is guessed to be in the state of qubit A after a successful projection onto the symmetric subspace of qubits A and B , while there is no guessing involved after a successful projection onto the antisymmetric subspace of qubits A and B . It is then known with certainty to be not in the state of qubit A .

The results for the minimum error probability, following from Eqs. (3.15) and (2.17), can be summarized as

$$P_E^{AB} = \begin{cases} \eta_1 & \text{if } \eta_1 \leq \frac{3}{7} \\ \frac{3}{4}(1 - \eta_1) & \text{otherwise.} \end{cases} \quad (3.16)$$

Similarly, a joint measurement on the qubits B and C yields the minimum error probability

$$P_E^{BC} = \begin{cases} \frac{3}{4}\eta_1 & \text{if } \eta_1 \leq \frac{4}{7} \\ 1 - \eta_1 & \text{otherwise.} \end{cases} \quad (3.17)$$

Figure 1 also reveals that by performing the optimal two-qubit measurement an error probability can be achieved that is almost as low as the absolute minimum error probability, P_E , given by Eq. (3.12) where the latter can only be reached by a joint measurement on all three qubits. Even when the advantage of the three-qubit measurement is largest, which happens for equal prior probabilities, $\eta_1 = \eta_2 = 1/2$, the difference in the respective minimum error probabilities for state identification is only marginal,

$$P_E^{AB} = P_E^{BC} = 0.375, \quad P_E = \frac{1}{2} - \frac{1}{4\sqrt{3}} = 0.356. \quad (3.18)$$

In the next paragraph we compare the minimum probabilities of error with the minimum probability of failure arising in the other important measurement strategy, that of unambiguous discrimination.

B. Optimum unambiguous discrimination strategy

The optimum measurement for unambiguously identifying the state of the data qubit B was found in Ref. [18] using a method that relied on a special ansatz for the detection operators, justified by the symmetry properties of the inputs. For completeness, here we reconsider the problem in the framework of the optimum unambiguous discrimination of two mixed states.

In the following we apply the method developed in Ref. [30] which, in turn, is a special case of the more general approach in Ref. [32]. Starting from Eqs. (3.4) and (3.5), we denote the projectors onto the supports of ρ_1 and ρ_2 by P_1 and P_2 , respectively. The eigenstates of the operators $I - P_2$ and $I - P_1$ are easily found to be

$$|a_1\rangle = \frac{|2\rangle - |3\rangle}{\sqrt{2}}, \quad |a_2\rangle = \frac{|6\rangle - |7\rangle}{\sqrt{2}}, \quad (3.19)$$

$$|b_1\rangle = \frac{|3\rangle - |4\rangle}{\sqrt{2}}, \quad |b_2\rangle = \frac{|5\rangle - |6\rangle}{\sqrt{2}}. \quad (3.20)$$

Clearly, $\rho_2|a_i\rangle = 0$ and $\rho_1|b_i\rangle = 0$ for $i=1,2$. The most general ansatz for the detection operators, satisfying $\Pi_1\rho_2 = \Pi_2\rho_1 = 0$ as required for unambiguous discrimination, therefore reads [30]

$$\Pi_1 = \sum_{i,j=1}^2 \alpha_{ij} |a_i\rangle \langle a_j|, \quad \Pi_2 = \sum_{i,j=1}^2 \beta_{ij} |b_i\rangle \langle b_j|. \quad (3.21)$$

From Eq. (2.19) we readily find that these two detection operators give rise to the failure probability

$$Q_{\text{fail}} = 1 - \frac{1}{8} \sum_{i=1}^2 (\eta_1 \alpha_{ii} + \eta_2 \beta_{ii}). \quad (3.22)$$

Note that due to the structure of the two given density operators the failure probability does not depend on the off-diagonal elements of the detection operators given by Eq. (3.21), a property that is common to all problems of optimum unambiguous discrimination of two mixed states that have been explicitly solved so far [29–32]. We are therefore free to choose $\alpha_{ij} = \beta_{ij} = 0$ for $i \neq j$, a choice that guarantees that Π_0 is positive for the largest possible values of α_{ii} and β_{ii} ($i=1,2$), i.e., that Q_{fail} can be made as small as possible.

When we represent the operator Π_0 in the basis defined in Eqs. (3.1), we again arrive at a block-diagonal eight by eight matrix, similar to Eq. (3.6), given by

$$\Pi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & M(\alpha_{11}, \beta_{11}) & 0 & 0 \\ 0 & 0 & M(\beta_{22}, \alpha_{22}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.23)$$

Here we introduced the abbreviation

$$M(a, b) = \frac{1}{2} \begin{pmatrix} 2-a & a & 0 \\ a & 2-a-b & b \\ 0 & b & 2-b \end{pmatrix}. \quad (3.24)$$

The eigenvalues of $M(a, b)$ are found to be 1 and

$$\mu_{\pm}(a, b) = 1 - \frac{1}{2}(a + b \pm \sqrt{(a-b)^2 + ab}), \quad (3.25)$$

where obviously $\mu_{\pm}(a, b) = \mu_{\pm}(b, a)$. All eigenvalues of Π_0 are nonnegative provided that $\mu_{+}(\alpha_{ii}, \beta_{ii})$ is nonnegative for $i=1, 2$ which holds true when $\beta_{ii} \leq (4-4\alpha_{ii})/(4-3\alpha_{ii})$. Hence in order to make Q_{fail} as small as possible, while keeping Π_0 a positive operator, we chose the equality sign and put

$$\beta_{ii} = \frac{4-4\alpha_{ii}}{4-3\alpha_{ii}} \quad (i=1, 2). \quad (3.26)$$

After inserting these expressions into Eq. (3.22) the resulting function of α_{11} and α_{22} has to be minimized, taking into account that $0 \leq \alpha_{ii} \leq 1$, which follows from the fact that $\text{Tr}(\rho\Pi_1)$ describes a probability for any density operator ρ . In accordance with the optimization problem solved in Ref. [18], we find that the failure probability takes its smallest possible value when $\alpha_{11} = \alpha_{22} = \alpha$, where

$$\alpha = \begin{cases} 0 & \text{if } \eta_1 \leq \frac{1}{4}\eta_2 \\ \frac{2}{3} \left(2 - \sqrt{\frac{\eta_2}{\eta_1}} \right) & \text{if } \frac{1}{4}\eta_2 \leq \eta_1 \leq 4\eta_2 \\ 1 & \text{if } \eta_1 \geq 4\eta_2. \end{cases} \quad (3.27)$$

Using Eqs. (3.21) and (3.1) we arrive at the optimum detection operators

$$\Pi_1^{\text{opt}} = \alpha I_A \otimes |\bar{u}_2\rangle_{BC} \langle \bar{u}_2|_{BC}, \quad (3.28)$$

$$\Pi_2^{\text{opt}} = \frac{4-4\alpha}{4-3\alpha} |\bar{u}_2\rangle_{AB} \langle \bar{u}_2|_{AB} \otimes I_C, \quad (3.29)$$

where the value of α in the different parameter regions for η_1 and η_2 is given by Eq. (3.27). Clearly, in the first parameter region $\Pi_1^{\text{opt}}=0$, while Π_2^{opt} describes a projection onto the antisymmetric two-qubit state $|\bar{u}_2\rangle_{AB}$. Similarly, in the third parameter region a projection onto the antisymmetric state $|\bar{u}_2\rangle_{BC}$ has to be performed for optimum unambiguous discrimination. The failure probabilities resulting from these two von Neumann measurements are [18]

$$Q_F^{AB} = 1 - \frac{\eta_2}{4} = \frac{3}{4} + \frac{\eta_1}{4}, \quad Q_F^{BC} = 1 - \frac{\eta_1}{4}. \quad (3.30)$$

In the intermediate region of the prior probabilities the optimum measurement is a generalized measurement, yielding the failure probability [18]

$$Q_F^{\text{POVM}} = \frac{2 + \sqrt{\eta_1(1-\eta_1)}}{3} \quad \left(\frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \right), \quad (3.31)$$

where we took into account that $\eta_2 = 1 - \eta_1$.

As seen in Fig. 1, the minimum failure probability is always at least twice as large as the minimum error probability P_E for identifying the qubit state. This agrees with the general relation between the failure probability of optimal unambiguous discrimination and the error probability of minimum-error discrimination of two mixed states that was derived in Ref. [35]. Figure 1 also shows that the advantage of performing a generalized measurement, as compared to the best projective two-qubit measurement, is small. For $\eta_1 = \eta_2 = 0.5$, where $Q_F^{AC} = Q_F^{BC} = 7/8 = 0.875$ the failure probability is only reduced to the value $Q_F^{\text{POVM}} = 5/6 = 0.833$. Of course, the surprise is not the high value of the failure probability but that the success probability is finite for the discrimination of completely unknown states. As we shall see in the next section, these relatively large failure probabilities are somewhat reduced when one of the reference states is known.

IV. ONE-QUBIT PROGRAM, SINGLE COPY OF THE DATA STATE

A. Minimum-error discrimination between one known and one unknown state

Now we treat the simplified case in which we want to decide whether the qubit B is in the known state $|0\rangle_B$, or whether it is in the unknown state of the program qubit stored in register C . We then have to distinguish between the density operators ρ'_1 and ρ'_2 given by Eqs. (2.8) and (2.9) that refer to the first and second alternative, respectively, and that occur with the prior probabilities η_1 and η_2 . The subsequent treatment proceeds along exactly the same lines that we followed in the previous sections.

For minimum-error identification we have to determine the eigenvalues and eigenstates of the operator $\Lambda' = \eta_2 \rho'_2 - \eta_1 \rho'_1$. It is easy to obtain the spectral representation

$$\Lambda' = \left(\frac{\eta_2}{3} - \frac{\eta_1}{2} \right) |u_1\rangle \langle u_1| + \frac{\eta_2}{3} |u_3\rangle \langle u_3| + \sum_{i=\pm} \lambda_i |\varphi_i\rangle \langle \varphi_i|, \quad (4.1)$$

where $\lambda_{\pm} = \frac{1}{12}(2\eta_2 - 3\eta_1 \pm \sqrt{4\eta_2^2 + 9\eta_1^2})$ and

$$|\varphi_{\pm}\rangle = \frac{|u_3\rangle - c_{\pm}|\bar{u}_2\rangle}{\sqrt{1+c_{\pm}^2}}, \quad c_{\pm} = \frac{2\eta_2 \pm \sqrt{4\eta_2^2 + 9\eta_1^2}}{3\eta_1}. \quad (4.2)$$

By making use of Eq. (2.17) we find that the minimum error probability for identifying the state of the qubit B is given by

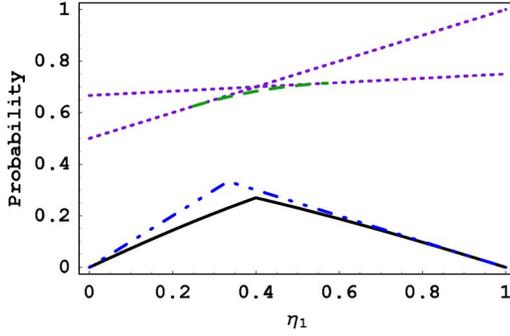


FIG. 2. (Color online) Failure and error probabilities for the various strategies of discriminating between one known and one unknown state vs the prior probability η_1 . The minimum error probabilities P'_E resulting from a joint measurement on the qubits B and C (full line) and from a single-qubit measurement, P''_E (dashed-dotted line) are compared to the failure probabilities for unambiguous identification Q'_F (dotted lines), and Q'_F (dashed line).

$$P'_E = \eta_{\min} \left(1 - \frac{1}{2} \frac{\eta_{\max}}{\eta_{\max} - \eta_{\min} + \sqrt{(\eta_{\max} - \eta_{\min})^2 + 2\eta_{\min}\eta_{\max}}} \right). \quad (4.3)$$

According to Eq. (2.16) it is reached with the help of the detection operators

$$\Pi_1^{\text{opt}} = \begin{cases} |\varphi_{-}\rangle_{BC} \langle \varphi_{-}|_{BC} & \text{if } \eta_1 \leq \frac{2}{5} \\ |u_1\rangle_{BC} \langle u_1|_{BC} + |\varphi_{-}\rangle_{BC} \langle \varphi_{-}|_{BC} & \text{otherwise,} \end{cases} \quad (4.4)$$

and $\Pi_2^{\text{opt}} = I_{BC} - \Pi_1^{\text{opt}}$, where we have made use of the identity $I_{BC} = |u_1\rangle \langle u_1| + |u_3\rangle \langle u_3| + \sum_{i=\pm} |\varphi_i\rangle \langle \varphi_i|$. Clearly, the measurement that identifies the state of the qubit B with the smallest possible error is a joint projection measurement on the qubits B and C .

To close this section we briefly investigate the case that we can only perform a measurement on the qubit B alone, which means that our identification problem amounts to discriminating between the density operator $\rho_1 = |0\rangle_{BB} \langle 0|$ and the uniformly mixed state $\rho_2 = \text{Tr}_C \rho_2 = \frac{1}{2} I_B$. For minimum-error identification we consider the eigenvalues and eigenstates of the operator $\Lambda = (\eta_2/2 - \eta_1)|0\rangle_{BB} \langle 0| + \eta_2/2|1\rangle_{BB} \langle 1|$, and obtain in the standard way, the minimum error probability

$$P''_E = \begin{cases} \eta_1 & \text{if } \eta_1 \leq \frac{1}{3} \\ \frac{1}{2}(1 - \eta_1) & \text{otherwise.} \end{cases} \quad (4.5)$$

If $\eta_1 \leq 1/3$ the error probability is smallest when the qubit B is always assumed to be in the unknown state of the qubit C , while otherwise a projection measurement characterized by the operators $\Pi_1^{\text{opt}} = |0\rangle_{BB} \langle 0|$ and $\Pi_2^{\text{opt}} = |1\rangle_{BB} \langle 1|$ has to be performed in order to minimize the error.

As can be seen from Fig. 2, for $\eta_1 \leq 2/5$ the optimum two-qubit measurement leads to a substantial reduction of

the minimum error probability in comparison to the optimum single-qubit measurement. The difference is largest for $\eta_1 = 1/3$, where $P'_E = 0.22$ but $P''_E = 0.33$.

B. Optimum unambiguous discrimination between one known and one unknown state

Finally we want to compare the minimum error probabilities with the minimum failure probability in unambiguously identifying the qubit state. For this purpose we again use the method described in Refs. [30,32]. By taking a reduction theorem [26] into account, from Eqs. (2.8) and (2.9) it follows that the most general ansatz for the detection operators can be written as

$$\Pi_1 = \alpha |\bar{u}_2\rangle \langle \bar{u}_2|, \quad \Pi_2 = \beta |\bar{v}_2\rangle \langle \bar{v}_2| + |u_3\rangle \langle u_3|, \quad (4.6)$$

where $|\bar{v}_2\rangle = (|u_2\rangle - |\bar{u}_2\rangle)/\sqrt{2}$. Here again the subscript BC has been dropped. Clearly, $\Pi_1 \rho_2 = \Pi_2 \rho_1 = 0$ as required for unambiguous discrimination. As follows from Eq. (2.19), these detection operators yield the failure probability

$$Q'_{\text{fail}} = 1 - \frac{\eta_1}{4} \alpha - \frac{\eta_2}{6} (2 + \beta) \quad (4.7)$$

which has to be minimized under the constraint that $\Pi_0 = I - \Pi_1 - \Pi_2$ is a positive operator. For Π_0 we obtain the expression

$$\Pi_0 = I - \alpha |\bar{u}_2\rangle \langle \bar{u}_2| - |u_3\rangle \langle u_3| - \frac{\beta}{2} (|u_2\rangle - |\bar{u}_2\rangle) (\langle u_2| - \langle \bar{u}_2|). \quad (4.8)$$

The eigenvalues of Π_0 are $\mu_1 = 1$, $\mu_2 = 0$, and $\mu_{\pm} = (2 - \beta - \alpha \pm \sqrt{\alpha^2 + \beta^2})/2$, and they all are nonnegative provided that $\beta \leq (2 - 2\alpha)/(2 - \alpha)$. In order to minimize Q'_{fail} while keeping Π_0 a positive operator we therefore choose

$$\beta = \frac{2 - 2\alpha}{2 - \alpha}. \quad (4.9)$$

Upon substituting β into Eq. (4.7) and determining the smallest value of the resulting function of α , taking into account that $0 \leq \alpha \leq 1$, we find that the minimum failure probability is obtained when

$$\alpha = \begin{cases} 0 & \text{if } 3\eta_1 \leq \eta_2 \\ 2 \left(1 - \sqrt{\frac{\eta_2}{3\eta_1}} \right) & \text{if } \eta_2 \leq 3\eta_1 \leq 4\eta_2 \\ 1 & \text{if } 3\eta_1 \geq 4\eta_2. \end{cases} \quad (4.10)$$

Using Eqs. (4.10) and (4.9) in Eq. (4.6) yields the explicit expressions for the optimum detection operators.

If $3\eta_1 \leq \eta_2$, which implies that $\eta_1 \leq 1/4$, we have $\Pi_1^{\text{opt}} = 0$ and $\Pi_2^{\text{opt}} = |1\rangle_{BB} \langle 1| \otimes I_C$, i.e., the optimum measurement is a projection measurement on the qubit B alone. On the other hand, for $3\eta_1 \geq 4\eta_2$, i.e., $\eta_1 \geq 4/7$, the optimum measurement is a joint projection measurement on the qubits B and C , where $\Pi_1^{\text{opt}} = |\bar{u}_2\rangle_{BC} \langle \bar{u}_2|$ and $\Pi_2^{\text{opt}} = |1\rangle_{BB} \langle 1| \otimes |1\rangle_C \langle 1|$. The failure probability of these two von Neumann measurements is given by

$$Q_F'^B = 1 - \frac{1 - \eta_1}{2},$$

$$Q_F'^{BC} = 1 - \frac{\eta_1}{4} - \frac{\eta_2}{3} = \frac{2}{3} + \frac{\eta_1}{12}. \quad (4.11)$$

In the intermediate parameter region the optimum measurement is a generalized measurement, yielding the failure probability

$$Q_F'^{\text{POVM}} = \frac{\eta_1}{6} + \frac{1 + \sqrt{3\eta_1(1-\eta_1)}}{3} \left(\frac{1}{4} \leq \eta_1 \leq \frac{4}{7} \right). \quad (4.12)$$

The benefit of performing the generalized measurement is only marginal, as evidenced by Fig. 2. In fact, the reduction of the failure probability compared to the best of the two types of von Neumann measurements is largest for $\eta_1=0.4$, where $Q_F'^B=Q_F'^{BC}=0.7$, while the generalized measurement yields the failure probability $Q_F'^{\text{POVM}}=0.683$. In agreement with the general relation derived in Ref. [35], the latter value is more than twice as large as the minimum error probability P_E' which for $\eta_1=0.4$ takes its maximum value 0.258.

A comparison of Figs. 1 and 2 reveals that, as expected, the minimum probabilities of error and failure are smaller when one reference state is known than in the case when both reference qubits are in unknown states.

V. TWO-QUBIT PROGRAM, n COPIES OF DATA STATE

A. Formulation of problem

We now return to the situation in which we possess only one copy of each of the two states we are trying to distinguish, but now we have $n > 1$ copies of the unknown state. This means that we want a POVM that will distinguish the two $n+2$ qubit states

$$|\Psi_1\rangle = |\psi_1\rangle_A \otimes |\psi_1\rangle_1 \otimes \cdots \otimes |\psi_1\rangle_n \otimes |\psi_2\rangle_C,$$

$$|\Psi_2\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_1 \otimes \cdots \otimes |\psi_2\rangle_n \otimes |\psi_2\rangle_C, \quad (5.1)$$

that occur with *a priori* probabilities η_1 and η_2 , respectively. Again we give two protocols, one for minimum error discrimination, and one for optimum unambiguous discrimination between the states.

To this end we now define the spaces and operators that we will need. Let Σ be the space of symmetric states in $\otimes^{n+1}\mathcal{H}$, where \mathcal{H} is the two-dimensional space for a single qubit. Σ is an $n+2$ dimensional subspace. $|\Psi_1\rangle$ is an element of $\Sigma \otimes \mathcal{H} = S_1$ and $|\Psi_2\rangle$ is an element of $\mathcal{H} \otimes \Sigma = S_2$. Their intersection, $S_0 = S_1 \cap S_2$, is the space of symmetric states in $\otimes^{n+2}\mathcal{H}$. S_0 is a subspace of dimension $n+3$. Let K be the subspace of $\otimes^{n+2}\mathcal{H}$ generated by S_1 and S_2 . The dimension of K is $3n+5$. Let S_3 be the orthogonal complement of S_0 in S_1 , let S_4 be the orthogonal complement of S_0 in S_2 , and let L be the orthogonal complement of S_0 in K . As was discussed in Sec. II, because we do not know what $|\psi_1\rangle$ and $|\psi_2\rangle$ are, our problem is to discriminate between the density matrices that result from averaging $|\Psi_1\rangle\langle\Psi_1|$ and $|\Psi_2\rangle\langle\Psi_2|$ over $|\psi_1\rangle$

and $|\psi_2\rangle$. This yields the two density matrices

$$\rho_1 = \frac{1}{2n+4} P_{S_1} = \frac{1}{2n+4} P_\Sigma \otimes I,$$

$$\rho_2 = \frac{1}{2n+4} P_{S_2} = \frac{1}{2n+4} I \otimes P_\Sigma, \quad (5.2)$$

where P_{S_1} and P_{S_2} are the projections onto S_1 and S_2 , and P_Σ and I onto Σ and \mathcal{H} , respectively. Consequently, we reduced the problem to discriminating between the $(2n+4)$ -dimensional spaces S_1 and S_2 in K , which is equivalent to discriminating between the $(n+1)$ -dimensional subspaces S_3 and S_4 in the $(2n+2)$ -dimensional space L .

We will now choose some bases in order to construct Jordan bases for these subspaces. Jordan bases $\{|p_j\rangle | j=0, \dots, N\}$ and $\{|r_j\rangle | j=0, \dots, N\}$ for two $(N+1)$ -dimensional subspaces, S_p and S_r , in general position, are orthonormal bases of their respective subspaces ($\{|p_j\rangle\}$ for S_p and $\{|r_j\rangle\}$ for S_r) that, in addition, satisfy $\langle r_j | p_k \rangle = \delta_{jk} \cos \theta_k$. The angles θ_k are called the Jordan angles. Now, let $|0\rangle$ and $|1\rangle$ be orthonormal basis vectors for \mathcal{H} . Further, let $|u_j^{(n+1)}\rangle$ ($j=0, \dots, n+1$) be the unique unit vector in the symmetric subspace of $n+1$ qubits, Σ , which is the sum of $(n+1)$ -tuples with j ones and $n-j+1$ zeros,

$$|u_0^{(n+1)}\rangle = |0 \dots 0\rangle,$$

$$|u_1^{(n+1)}\rangle = \frac{|0 \dots 01\rangle + |0 \dots 10\rangle + \cdots + |10 \dots 0\rangle}{\sqrt{n+1}},$$

$$\vdots$$

$$|u_{n+1}^{(n+1)}\rangle = |11 \dots 1\rangle. \quad (5.3)$$

Then the structure of the two density operators in (5.2), in particular the decomposition on the right-hand side, suggests that we consider $|e_{j,\alpha}\rangle = |u_j^{(n+1)}\rangle \otimes |\alpha\rangle$ and $|f_{j,\alpha}\rangle = |\alpha\rangle \otimes |u_j^{(n+1)}\rangle$, where $\alpha=0, 1$ and $0 \leq j \leq n+1$. These vectors form orthonormal bases for S_1 and S_2 , respectively. Let $|u_j^{(n+2)}\rangle$ ($j=0, \dots, n+2$) be the unique unit vector in the symmetric subspace of $n+2$ qubits, S_0 , which is the sum of $(n+2)$ -tuples with exactly j ones and $n+2-j$ zeros. This vector can be expressed in terms of either the S_0 or S_1 basis, since it is in both spaces. A direct calculation shows that

$$|u_j^{(n+2)}\rangle = \sqrt{\frac{n+2-j}{n+2}} |e_{j,0}\rangle + \sqrt{\frac{j}{n+2}} |e_{j-1,1}\rangle, \quad (5.4)$$

and

$$|u_j^{(n+2)}\rangle = \sqrt{\frac{n+2-j}{n+2}} |f_{j,0}\rangle + \sqrt{\frac{j}{n+2}} |f_{j-1,1}\rangle \quad (5.5)$$

for $0 \leq j \leq n+2$. In particular, $|u_0^{(n+2)}\rangle = |00 \dots 0\rangle$ and $|u_{n+2}^{(n+2)}\rangle = |11 \dots 1\rangle$.

We now introduce the vectors

$$|g_j\rangle = \sqrt{\frac{j}{n+2}}|e_{j,0}\rangle - \sqrt{\frac{n+2-j}{n+2}}|e_{j-1,1}\rangle, \quad (5.6)$$

and

$$|h_j\rangle = \sqrt{\frac{j}{n+2}}|f_{j,0}\rangle - \sqrt{\frac{n+2-j}{n+2}}|f_{j-1,1}\rangle, \quad (5.7)$$

for $1 \leq j \leq n+1$. The $|g_j\rangle$'s and $|h_j\rangle$'s form orthonormal bases for S_3 and S_4 . Each vector on the right-hand side of the above expressions has exactly j ones. Therefore, if $j \neq k$

$$\langle g_j | h_k \rangle = 0, \quad (5.8)$$

and $\{|g_j\rangle\}$ and $\{|h_j\rangle\}$ form Jordan bases for S_3 and S_4 . Let T_j be the two-dimensional vector space spanned by the nonorthogonal but linearly independent vectors $|g_j\rangle$ and $|h_j\rangle$. The T_j form a decomposition of L into $n+1$ mutually perpendicular two-dimensional subspaces. A calculation shows that

$$\begin{aligned} \langle f_{j,0} | e_{j,0} \rangle &= \frac{n+1-j}{n+1}, \\ \langle f_{j-1,1} | e_{j-1,1} \rangle &= \frac{j-1}{n+1}, \end{aligned} \quad (5.9)$$

and

$$\langle e_{j-1,1} | f_{j,0} \rangle = \langle f_{j-1,1} | e_{j,0} \rangle = \frac{(j(n+2-j))^{1/2}}{n+1}. \quad (5.10)$$

Therefore,

$$\langle h_j | g_j \rangle = -\frac{1}{n+1}, \quad (5.11)$$

and the Jordan angles are all the same. The two density operators that we wish to distinguish can now be expressed as

$$\begin{aligned} \rho_1 &= \frac{1}{2(n+2)} \left[P_{S_0} + \sum_{i=1}^{n+1} |g_i\rangle\langle g_i| \right], \\ \rho_2 &= \frac{1}{2(n+2)} \left[P_{S_0} + \sum_{i=1}^{n+1} |h_i\rangle\langle h_i| \right], \end{aligned} \quad (5.12)$$

where

$$P_{S_0} = \sum_{j=0}^{n+2} |u_j^{(n+2)}\rangle\langle u_j^{(n+2)}| \quad (5.13)$$

is the projection onto S_0 .

B. Minimum error discrimination strategy

For minimum-error identification we have to determine the eigenvalues and eigenstates of the operator $\Lambda = \eta_2 \rho_2 - \eta_1 \rho_1$ which, using Eq. (5.12), can be written as

$$\Lambda = \frac{1}{2n+4} \left[(\eta_2 - \eta_1) P_{S_0} + \eta_2 \sum |h_i\rangle\langle h_i| - \eta_1 \sum |g_i\rangle\langle g_i| \right]. \quad (5.14)$$

Λ is diagonal in S_0 and it is straightforward to carry out the diagonalization in each of the two-dimensional subspaces spanned by $|g_i\rangle$ and $|h_i\rangle$, yielding the spectral representation,

$$\Lambda = \sum_{j=0}^{n+2} \lambda_0 |u_j\rangle\langle u_j| + \sum_{i=1}^{n+1} (\lambda_+ |\varphi_{i+}\rangle\langle \varphi_{i+}| + \lambda_- |\varphi_{i-}\rangle\langle \varphi_{i-}|), \quad (5.15)$$

where

$$\lambda_0 = \frac{\eta_2 - \eta_1}{2n+4}, \quad (5.16)$$

and

$$\lambda_{\pm} = \frac{1}{4(n+2)} \left(\eta_2 - \eta_1 \pm \sqrt{(\eta_2 - \eta_1)^2 + \frac{4n(n+2)}{(n+1)^2} \eta_1 \eta_2} \right). \quad (5.17)$$

The eigenvalue associated with S_0 , λ_0 , has a degeneracy $(n+3)$ and the eigenvalues associated with S_3 and S_4 , λ_{\pm} , have a degeneracy of $(n+1)$ each. Furthermore, in the nonorthogonal basis of the two-dimensional subspace T_i given by $|g_i\rangle$ and $|h_i\rangle$,

$$|\varphi_{i\pm}\rangle = \frac{|g_i\rangle - c_{\pm}|h_i\rangle}{\sqrt{1 + c_{\pm}^2 - 2c_{\pm}/(n+1)}}, \quad (5.18)$$

where

$$c_{\pm} = (n+1) \left[1 - \frac{\eta_2}{(n+1)^2} - \eta_1 \right] \frac{1}{(2n+4)\lambda_{\pm}}. \quad (5.19)$$

We find that λ_- is unconditionally negative, λ_0 is negative if $\eta_2 < \eta_1$ and positive otherwise, and λ_+ is unconditionally positive. By making use of Eq. (2.17) we find that the minimum error probability for identifying the state of the data qubits is given by

$$P_E = \eta_{\min} \left[1 - \frac{n}{n+1} \frac{\eta_{\max}}{\eta_{\max} - \eta_{\min} + \sqrt{(\eta_{\max} - \eta_{\min})^2 + \frac{4n(n+2)}{(n+1)^2} \eta_{\min} \eta_{\max}}} \right], \quad (5.20)$$

where η_{\min} (η_{\max}) is the smaller (larger) of $\{\eta_1, \eta_2\}$. According to Eq. (2.16), the minimum error probability is reached with the help of the optimum detection operators

$$\Pi_1^{\text{opt}} = \begin{cases} \sum_{i=1}^{n+1} |\varphi_{i-}\rangle\langle\varphi_{i-}| & \text{if } \eta_1 \leq \frac{1}{2} \\ P_{S_0} + \sum_{i=1}^{n+1} |\varphi_{i-}\rangle\langle\varphi_{i-}| & \text{if } \eta_1 > \frac{1}{2} \end{cases}, \quad (5.21)$$

which is the projection onto the strictly negative eigenspace of Λ , and $\Pi_2^{\text{opt}} = I_K - \Pi_1^{\text{opt}}$, where the identity $I_K = P_{S_0} + \sum_{i=1}^{n+1} (|\varphi_{i+}\rangle\langle\varphi_{i+}| + |\varphi_{i-}\rangle\langle\varphi_{i-}|)$. P_{S_0} is given in Eq. (5.13). Clearly, the measurement that identifies the state of the data qubits with the smallest possible error is a joint projection measurement on all of the qubits. It should be noted that for $n=1$ the formulas in this section reduce to those of Sec. III A, whereas for $n \rightarrow \infty$ we have that $P_E \rightarrow \eta_{\min}/2 \leq 1/4$.

C. The optimal universal unambiguous bound

We now want to consider the unambiguous discrimination between the subspaces S_1 and S_2 in K , or equivalently between S_3 and S_4 in L . Let S_i^\perp be the orthogonal complement of S_i in K . S_i^\perp is equal to the orthogonal complement of S_{i+2} in L . S_i^\perp is an $(n+1)$ -dimensional subspace. The POVM which unambiguously distinguishes between S_1 and S_2 has the form $\Pi_1 = \alpha P_{S_2^\perp}$, and $\Pi_2 = \beta P_{S_1^\perp}$, where the P 's are orthogonal projections onto S_1^\perp or S_2^\perp , and the α and β are positive real numbers between zero and one, which are so chosen that Π_1, Π_2 , and $\Pi_0 = I - \Pi_1 - \Pi_2$ are the elements of a POVM on K .

Let us define $|g_i^\perp\rangle$ in S_1^\perp , and $|h_i^\perp\rangle$ in S_2^\perp by the formulas

$$\begin{aligned} |h_i^\perp\rangle &= -\frac{1}{n+1}|g_i\rangle + \frac{\sqrt{n(n+2)}}{n+1}|g_i^\perp\rangle, \\ |g_i^\perp\rangle &= -\frac{1}{n+1}|h_i\rangle + \frac{\sqrt{n(n+2)}}{n+1}|h_i^\perp\rangle, \end{aligned} \quad (5.22)$$

on T_i , and we have that

$$P_{S_1^\perp} = \sum_{i=1}^{n+1} |g_i^\perp\rangle\langle g_i^\perp|, \quad P_{S_2^\perp} = \sum_{i=1}^{n+1} |h_i^\perp\rangle\langle h_i^\perp|. \quad (5.23)$$

The α and β can now be chosen so that Π_0 restricted to each T_i is nonnegative. The matrix which represents Π_0 on T_i , in the basis $\{|g_i\rangle, |g_i^\perp\rangle\}$, is

$$\begin{pmatrix} 1 - \alpha \frac{n^2 + 2n}{(n+1)^2} & -\alpha \frac{\sqrt{n(n+n)}}{(n+1)^2} \\ -\alpha \frac{\sqrt{n(n+n)}}{(n+1)^2} & 1 - \beta - \frac{\alpha}{(n+1)^2} \end{pmatrix}. \quad (5.24)$$

This matrix must be positive. Therefore, Π_1, Π_2 , and Π_0 form a POVM if and only if

$$1 - \alpha - \beta - \alpha\beta \frac{n^2 + 2n}{(n+1)^2} \geq 0, \quad (5.25)$$

provided that

$$0 \leq \alpha, \beta \leq 1. \quad (5.26)$$

We also have that

$$\text{Tr}(\Pi_1 \rho_1) = \frac{\alpha n}{2n+2}, \quad \text{Tr}(\Pi_2 \rho_2) = \frac{\beta n}{2n+2}. \quad (5.27)$$

If we are given ρ_1 with probability η_1 and ρ_2 with probability η_2 then the probability that this POVM successfully distinguishes S_1 from S_2 is

$$P_{\text{succ}} = \frac{n}{2n+2} (\eta_1 \alpha + \eta_2 \beta), \quad (5.28)$$

if the α and β satisfy the constraints above. If we set

$$\beta = \frac{1 - \alpha}{1 - \frac{n^2 + 2n}{(n+1)^2} \alpha} \quad (5.29)$$

in the above formula, which is the maximum allowed by (5.25), and differentiate, we find that P_{succ} has a maximum value when

$$\alpha = \begin{cases} 0 & \text{if } \eta_1 \leq \frac{1}{1 + (n+1)^2} \\ \frac{n+1}{n^2+n} \left(n+1 - \sqrt{\frac{\eta_2}{\eta_1}} \right) & \text{if } \frac{1}{1 + (n+1)^2} \leq \eta_1 \leq \frac{(n+1)^2}{1 + (n+1)^2} \\ 1 & \text{if } \eta_1 \geq \frac{(n+1)^2}{1 + (n+1)^2}. \end{cases} \quad (5.30)$$

The maximum value of P_{succ} is

$$P_{\text{max}} = \frac{1}{n+2} \left[\frac{n+1}{2} - (\eta_1 \eta_2)^{1/2} \right], \quad (5.31)$$

provided $\frac{1}{1+(n+1)^2} \leq \eta_1 \leq \frac{(n+1)^2}{1+(n+1)^2}$, using the center line in (5.30). This clearly can be obtained by a POVM only. If η_1 is to the left of this interval [first line in (5.30)] the optimum measurement is the projection $P_{S_1^\perp}$, which unambiguously identifies ρ_2 with a success probability $P_{\text{succ}}^{(2)} = \eta_2 n / (2n+2)$. If η_1 is to the right of this interval [last line in (5.30)] the optimum measurement is the projection $P_{S_2^\perp}$ which unambiguously identifies ρ_1 with a success probability $P_{\text{succ}}^{(1)} = \eta_1 n / (2n+2)$. It should be noted that the optimum failure probability is given as $Q_F = 1 - P_{\text{succ}}$. For $n=1$ these expressions reproduce the corresponding ones in Sec. III B. For $\eta_1 = \eta_2 = 1/2$, when their difference is the largest, $Q_F^{\text{POVM}} = 1 - P_{\text{max}} = \frac{n+4}{2n+4}$ and $Q_F^{(1,2)} = 1 - P_{\text{succ}}^{(1,2)} = \frac{3n+4}{4n+4}$, as a function of n . For $n \rightarrow \infty$ we have $Q_F^{\text{POVM}} \rightarrow 1/2$ and $Q_F^{(1,2)} \rightarrow 3/4$, so the POVM outperforms the projective measurements quite significantly. Furthermore, $P_E \leq Q_F^{\text{POVM}}/2$ always holds, as it should.

If we use the universal POVM Π_1 , Π_2 , and Π_0 to unambiguously discriminate between the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ without averaging over them, we find that the probability of success, \tilde{P}_{succ} is

$$\begin{aligned} \tilde{P}_{\text{succ}} &= \eta_1 \alpha \langle \Psi_1 | \Pi_1 | \Psi_1 \rangle + \eta_2 \beta \langle \Psi_2 | \Pi_2 | \Psi_2 \rangle \\ &= \frac{n}{n+1} (\eta_1 \alpha + \eta_2 \beta) (1 - |\langle \psi_1 | \psi_2 \rangle|^2), \end{aligned} \quad (5.32)$$

where the α and β satisfy the same constraints as above. Inserting their optimal values from Eqs. (5.30) and (5.29) we find

$$\tilde{P}_{\text{opt}} = \frac{1 - |\langle \psi_1 | \psi_2 \rangle|^2}{n+2} [n+1 - 2(\eta_1 \eta_2)^{1/2}], \quad (5.33)$$

with the same restrictions on η_1 as in the previous paragraph. If we average the overlap term over all possible choices of the $|\psi_i\rangle$'s we can replace it with its average value of $1/2$, and we recover (5.31).

As expected, the optimal success probability is an increasing function of n . The more copies of the unknown qubit we possess, the greater our chance of identifying it.

The $n \rightarrow \infty$ limit of \tilde{P}_{opt} can also be achieved by a different strategy than the one we are employing here. With a very large number of copies of the unknown qubit, we could employ state reconstruction techniques to find out its state [36]. For example, suppose we have determined the state of the unknown qubit to be $|\psi_0\rangle$. While we know what this state is, we do not know if it is equal to the first or the second program state, $|\psi_1\rangle$ or $|\psi_2\rangle$. We therefore project each of the two program states onto the state orthogonal to the reconstructed state. That is, we take the program qubit that we know is in the state $|\psi_i\rangle$ ($i=1$ or 2) and measure the projection operator $P_{0^\perp} = |\psi_0^\perp\rangle\langle\psi_0^\perp|$. If $|\psi_0\rangle = |\psi_1\rangle$, which is given with the *a priori* probability η_1 , then this measurement succeeds (gives 1)

with a probability of $\langle \psi_1^\perp | \psi_2 \rangle|^2$, and if $|\psi_0\rangle = |\psi_2\rangle$, which is given with the *a priori* probability η_2 , it succeeds with the same probability. Therefore, the total probability of success is just $(\eta_1 + \eta_2) \langle \psi_1^\perp | \psi_2 \rangle|^2$, which is the same as the $n \rightarrow \infty$ limit of Eq. (5.33). While these strategies give the same result for an infinite number of copies, there is a difference between them for n finite. The strategy that led to Eq. (5.33) will never produce an erroneous result, while the strategy based on state reconstruction can. If our determination of the state of the unknown qubit is not exact, which will, in general, be the case for finite n , then $|\psi_0^\perp\rangle$ will not be orthogonal to either $|\psi_1\rangle$ or $|\psi_2\rangle$, and this will lead to errors.

VI. ONE-QUBIT PROGRAM, n COPIES OF DATA STATE

In this section we return to the case when one of the states to be determined is known and the other is unknown. However, in contrast to the treatment that was presented in Sec. IV, we now provide n copies of the states to be determined. In other words, we now have n data registers, B_1, \dots, B_n , and in each one we either have a copy of the known state $|0\rangle$ or a copy of the unknown state $|\psi\rangle$ stored in the program register C . Our task is then to decide whether the qubit B_i is in the known state $|0\rangle_{B_i}$, or whether it is in the unknown state of the program qubit stored in register C , for all i .

Thus, we assume that we have a system of $n+1$ qubits, labeled B_1, \dots, B_n , and C where C is the program qubit and B_1, \dots, B_n are the data qubits. Qubit C is always prepared in the state $|\psi\rangle$. Qubits B_i , the data qubits, are guaranteed to be all prepared in either $|0\rangle$ or all in $|\psi\rangle$, but we do not know which of these two alternatives occurs. The prior probabilities of these two alternatives are η_1 and $\eta_2 = 1 - \eta_1$, respectively. Our task is then to find the optimal measurement (POVM) that will distinguish the two $n+1$ qubit states,

$$\begin{aligned} |\Psi_1\rangle &= |0\rangle_1 \otimes \dots \otimes |0\rangle_n \otimes |\psi\rangle_C, \\ |\Psi_2\rangle &= |\psi\rangle_1 \otimes \dots \otimes |\psi\rangle_n \otimes |\psi\rangle_C, \end{aligned} \quad (6.1)$$

where we dropped the subscript B for the data registers as it leads to no confusion.

If the state $|\psi\rangle$ is completely unknown, we have to find the best measurement strategy that is optimal on average. Thus, we have to take the average of the input with respect to all possible qubit states. The identification problem is then equivalent to distinguishing between two mixed states, given by the density operators

$$\rho'_1 = \{ |\Psi_1\rangle\langle\Psi_1| \}_{\text{av}} = \{ |0\rangle^{\otimes n} \langle 0| \otimes \langle \psi| \}_{\text{av}}, \quad (6.2)$$

$$\rho'_2 = \{ |\Psi_2\rangle\langle\Psi_2| \}_{\text{av}} = \{ |\psi\rangle^{\otimes n+1} \langle \psi| \}_{\text{av}}, \quad (6.3)$$

that occur with the prior probabilities η_1 and η_2 , respectively. The unknown qubit state can be represented again using the Bloch parametrization as $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi} \sin(\theta/2)|1\rangle$, with $|0\rangle$ and $|1\rangle$ denoting an arbitrary set of orthonormal basis states. Here θ and ϕ are the polar and azimuthal angles on the Bloch sphere. After performing the averaging with respect to all possible values of θ and ϕ we arrive at

$$\rho'_1 = \frac{1}{2} |0\rangle^{\otimes(n)} \langle 0|^{\otimes(n)} \otimes I_C, \quad (6.4)$$

$$\rho'_2 = \frac{1}{(n+2)} P_{B_1 \cdots B_n C}^{\text{sym}}, \quad (6.5)$$

where $P_{B_1 \cdots B_n C}^{\text{sym}}$ is the projector to the $(n+2)$ -dimensional symmetric subspace of the corresponding $n+1$ qubits, $B_1 \cdots B_n C$, and $I_C = |0\rangle_{CC} \langle 0| + |1\rangle_{CC} \langle 1|$. Equations (6.4) and (6.5) reduce the problem of the programmable state discriminator with the single-qubit program and n copies of the data state to the problem of discriminating between these two mixed states.

At this point it is useful to introduce the following basis for the $(n+2)$ -dimensional symmetric subspace of the $n+1$ qubits

$$\begin{aligned} |u_1\rangle &= |0 \dots 0\rangle, \\ |u_2\rangle &= \frac{|0 \dots 01\rangle + |0 \dots 10\rangle + \dots + |10 \dots 0\rangle}{\sqrt{n+1}}, \\ &\vdots \\ |u_{n+2}\rangle &= |11 \dots 1\rangle. \end{aligned} \quad (6.6)$$

We also introduce

$$|v_2\rangle = |0 \dots 01\rangle. \quad (6.7)$$

The two nonorthogonal but linearly independent vectors, $|u_2\rangle$ and $|v_2\rangle$, span a two-dimensional subspace of the entire Hilbert space. It will prove useful later on to define two other vectors in this subspace as

$$|\bar{v}_2\rangle = \frac{|0 \dots 10\rangle + \dots + |10 \dots 0\rangle}{\sqrt{n}}. \quad (6.8)$$

and

$$|\bar{u}_2\rangle = \frac{1}{\sqrt{n+1}} |\bar{v}_2\rangle - \sqrt{\frac{n}{n+1}} |v_2\rangle. \quad (6.9)$$

The two sets, $\{|v_2\rangle, |\bar{v}_2\rangle\}$ and $\{|u_2\rangle, |\bar{u}_2\rangle\}$, each form an orthonormal basis in the two-dimensional subspace. We then have to distinguish between the density operators ρ'_1 and ρ'_2 given by Eqs. (6.4) and (6.5) that refer to the first and second alternative, respectively, and that occur with the prior probabilities η_1 and η_2 . After re-expressing ρ'_1 in terms of the basis states $|u_i\rangle$ ($i=1, \dots, n+2$) and $|v_2\rangle$, defined by Eqs. (6.6) and (6.7), respectively, the density operators to be discriminated read

$$\rho'_1 = \frac{1}{2} (|u_1\rangle \langle u_1| + |v_2\rangle \langle v_2|), \quad (6.10)$$

$$\rho'_2 = \frac{1}{n+2} \left(\sum_{i=1}^{n+2} |u_i\rangle \langle u_i| \right). \quad (6.11)$$

The subsequent treatment proceeds exactly along the same lines that we followed in the previous sections.

A. Minimum-error discrimination

For identifying the data state with minimum error we have to determine the eigenvalues and eigenstates of the operator $\Lambda' = \eta_2 \rho'_2 - \eta_1 \rho'_1$, where ρ'_1 and ρ'_2 are given by Eqs. (6.4) and (6.5). From the explicit expression of these two density operators it is clear that Λ' is diagonal except in the two-dimensional subspace spanned by $|v_2\rangle$ and $|\bar{v}_2\rangle$. It is straightforward to carry out the diagonalization in this subspace yielding the spectral representation,

$$\Lambda' = \lambda_1 |u_1\rangle \langle u_1| + \sum_{i=-}^{+} \lambda_i |\varphi_i\rangle \langle \varphi_i| + \sum_{j=3}^{n+2} \lambda_j |u_j\rangle \langle u_j|, \quad (6.12)$$

where

$$\lambda_1 = \left(\frac{\eta_2}{n+2} - \frac{\eta_1}{2} \right), \quad (6.13)$$

$$\lambda_{\pm} = \frac{1}{2} \left(\frac{\eta_2}{n+2} - \frac{\eta_1}{2} \pm \sqrt{\left(\frac{\eta_2}{n+2} - \frac{\eta_1}{2} \right)^2 + \frac{2\eta_1\eta_2 n}{(n+1)(n+2)}} \right), \quad (6.14)$$

and

$$\lambda_j = \frac{\eta_2}{n+2}, \quad (6.15)$$

for $j \geq 3$. Furthermore

$$|\varphi_{\pm}\rangle = \frac{|v_2\rangle - c_{\pm} |\bar{v}_2\rangle}{\sqrt{1 + c_{\pm}^2}}, \quad (6.16)$$

where

$$c_{\pm} = \frac{n+1}{2\sqrt{n}} \frac{\eta'_2 + \frac{n-1}{2(n+1)} \eta_1 \pm \sqrt{(\eta'_2 - \eta_1)^2 + \frac{4n}{n+1} \eta'_2 \eta_1}}{\eta_1}. \quad (6.17)$$

Here we introduced $\eta'_2 = 2\eta_2/(n+2)$ which is the weight of ρ'_2 in the intersection of the supports of the two density operators to be discriminated. We find that λ_- is unconditionally negative, λ_1 is negative if $\eta_2/(n+2) < \eta_1/2$ and positive otherwise, and $\lambda_+, \lambda_j \geq 0$ for $j \geq 3$. By making use of Eq. (2.17) we find that the minimum error probability for identifying the state of the data qubits is given by

$$P'_E = \eta_{\min} \left[1 - \frac{n}{n+1} \frac{\eta_{\max}}{\eta_{\max} - \eta_{\min} + \sqrt{(\eta_{\max} - \eta_{\min})^2 + \frac{4n\eta_{\min}\eta_{\max}}{n+1}}} \right], \quad (6.18)$$

where we introduced η_{\min} (η_{\max}) as the smaller (larger) of $\{\eta_1, 2\eta_2/(n+2)\}$. According to Eq. (2.16), the minimum error probability is reached with the help of the detection operators

$$\Pi_1^{\text{opt}} = \begin{cases} |\varphi_-\rangle\langle\varphi_-| & \text{if } \eta_1 \leq \frac{2}{n+4} \\ |u_1\rangle\langle u_1| + |\varphi_-\rangle\langle\varphi_-| & \text{if } \eta_1 > \frac{2}{n+4} \end{cases}, \quad (6.19)$$

and $\Pi_2^{\text{opt}} = I - \Pi_1^{\text{opt}}$, where we have to use the identity $I = |u_1\rangle\langle u_1| + \sum_{i=3}^{n+2} |u_i\rangle\langle u_i| + \sum_{i=-}^+ |\varphi_i\rangle\langle\varphi_i|$. Clearly, the measurement that identifies the state of the data qubits with the smallest possible error is a joint projection measurement on the qubits B_i (for $i=1, \dots, n$) and C . It should be noted that for $n=1$ the formulas in this section reduce to those of Sec. IV A; whereas for $n \rightarrow \infty$ we have that $P_E \rightarrow 0$ since in this latter case the mixed states that we are trying to distinguish become essentially orthogonal. The vanishing of the error probability for $n \rightarrow \infty$ is in accordance with the fact that the data state can be in principle exactly determined by tomographic methods, without any joint measurement, provided that an infinite number of copies is available. After the data state has been determined, it is of course possible to tell without error whether it is equal to the state $|0\rangle$ or not.

B. Unambiguous discrimination

Finally we want to determine the minimum failure probability for the unambiguous discrimination between the states given by (6.10) and (6.11). For this purpose we again use the method described in Refs. [30,32]. Taking one of the reduction theorems derived in Ref. [26] into account, the

most general ansatz for the detection operators can be written as

$$\Pi_1 = \alpha |\bar{u}_2\rangle\langle\bar{u}_2|, \quad \Pi_2 = \beta |\bar{v}_2\rangle\langle\bar{v}_2| + \sum_{i=3}^{n+2} |u_i\rangle\langle u_i|, \quad (6.20)$$

where $|\bar{v}_2\rangle$ and $|\bar{u}_2\rangle$ were given in (6.8) and (6.9), respectively. Clearly, $\Pi_1 \rho_2' = \Pi_2 \rho_1' = 0$ as required for unambiguous discrimination. As follows from Eq. (2.19), these detection operators yield the failure probability

$$Q'_{\text{fail}} = 1 - \frac{\eta_1 \alpha n}{2(n+1)} - \frac{\eta_2 n}{n+2} - \frac{\eta_2 \beta n}{(n+1)(n+2)} \quad (6.21)$$

which again has to be minimized under the constraint that $\Pi_0 = I - \Pi_1 - \Pi_2$ is a positive operator, in complete analogy to our procedure in Sec. IV A. For Π_0 we obtain the expression

$$\Pi_0 = |u_1\rangle\langle u_1| - \alpha |\bar{u}_2\rangle\langle\bar{u}_2| + |v_2\rangle\langle v_2| - \beta |\bar{v}_2\rangle\langle\bar{v}_2| + |\bar{v}_2\rangle\langle\bar{v}_2|. \quad (6.22)$$

The eigenvalues of Π_0 are $\mu_1=1, \mu_i=0$ for $3 \leq i \leq n+2$, and $\mu_{\pm} = (2 - \beta - \alpha \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta n/(n+1)})/2$. They all are nonnegative provided that $\beta \leq (n+1)(1-\alpha)/(n+1-n\alpha)$. In order to minimize Q'_{fail} while keeping Π_0 a positive operator we therefore choose

$$\beta = \frac{(n+1)(1-\alpha)}{n+1-n\alpha}. \quad (6.23)$$

Upon substituting this expression into Eq. (6.21) the failure probability becomes a function of α alone and it is easy to determine its optimum. Taking into account that $0 \leq \alpha \leq 1$, we find that the minimum failure probability is obtained when

$$\alpha_{\text{opt}} = \begin{cases} 0 & \text{if } (n+2)\eta_1 \leq 2\eta_2/(n+1) \\ \frac{n+1}{n} \left(1 - \sqrt{\frac{2\eta_2}{(n+1)(n+2)\eta_1}} \right) & \text{if } \frac{2\eta_2}{n+1} \leq (n+2)\eta_1 \leq 2(n+1)\eta_2 \\ 1 & \text{if } (n+2)\eta_1 \geq 2(n+1)\eta_2. \end{cases} \quad (6.24)$$

Substituting these values into (6.23) yields β_{opt} . Using α_{opt} and β_{opt} in Eqs. (6.20) and (6.22) yields an explicit expression for the optimum detection operators.

If $(n+2)\eta_1 \leq 2\eta_2/(n+1)$, which implies that $\eta_1 \leq 2/[2 + (n+1)(n+2)]$, we have $\Pi_1^{\text{opt}} = 0$ and $\Pi_2^{\text{opt}} = |\bar{v}_2\rangle\langle\bar{v}_2|$

+ $\sum_{i=3}^{n+2} |u_i\rangle\langle u_i|$, which means that the optimum measurement is a projection measurement on the subspace orthogonal to the span of ρ_1' , i.e., a projection on its kernel. On the other hand, for $(n+2)\eta_1 \geq 2(n+1)\eta_2$, i.e., $\eta_1 \geq 2(n+1)/(3n+4)$, the optimum measurement is a joint projection measurement on the

kernels of ρ'_1 and ρ'_2 , where $\Pi_1^{\text{opt}} = |\bar{u}_2\rangle\langle\bar{u}_2|$ and $\Pi_2^{\text{opt}} = \sum_{i=3}^{n+2} |u_i\rangle\langle u_i|$. In the intermediate parameter region the optimum measurement is a generalized measurement. The failure probability of these optimal measurements can be summarized as

$$Q'_F = \begin{cases} \eta_1 + \frac{\eta_2}{n+1} & \text{if } \eta_1 \leq \frac{2}{2+(n+1)(n+2)} \\ \frac{\eta_1}{2} + \frac{\eta_2}{n+2} + \sqrt{\frac{2\eta_1\eta_2}{(n+1)(n+2)}} & \text{if } \frac{2}{2+(n+1)(n+2)} \leq \eta_1 \leq \frac{2(n+1)}{2(n+1)+(n+2)} \\ \eta_1 \frac{n+2}{2n+2} + \eta_2 \frac{2}{n+2} & \text{if } \eta_1 \geq \frac{2(n+1)}{2(n+1)+(n+2)}. \end{cases} \quad (6.25)$$

We notice the above expressions reduce to the corresponding expressions of Sec. IV A for $n=1$, as they should. Then, as in that section, it is also true here that the benefit of performing the generalized measurement is only marginal. To see the closeness of the best projective valued measurement (PVM) to the optimal POVM we compare their performance in several ways. The two PVMs [first and last line in (6.25)] deliver the same result for $\eta_1=2/(n+4)$. In fact, the reduction of the POVM failure probability (middle line) compared to those of the projective measurements is largest for this value of η_1 . In Fig. 3 we display the PVM and POVM failure probabilities for this value of η_1 as a function of n . We see that the two curves remain close together for all values of n . The difference between these two curves as a function of n reaches a maximum, however. It is maximal for $n=5$ as displayed in Fig. 4. Finally, in Fig. 5 we display the ratio of the PVM failure probability to the POVM failure probability as a function of n . Asymptotically, the POVM outperforms the PVM by 50%, their ratio tending to the limiting value of 1.5. However, as we see from the figure, one needs about 1000 copies of the data state to reach the asymptotic region. Since the difference is maximal for five copies we can conclude that one does not need more than five copies in order to demonstrate performance enhancement due to the optimal POVM.

To close this section we also notice that, in agreement with the general relation derived in Ref. [35], the optimal

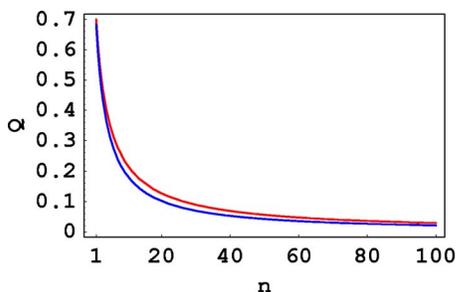


FIG. 3. (Color online) Comparison of the optimum performances of the failure probabilities of the projective measurements (upper curve) and the failure probability of the POVM (lower curve) vs the number of copies n for the value of $\eta_1=2/(n+4)$, when their difference is the largest for the unambiguous discrimination of one known state from one unknown state when n copies of the data state are provided.

POVM failure probability is always more than twice as large as the minimum error probability P'_E of the previous subsection.

VII. CONCLUSION

We have described a number of quantum devices that discriminate between two quantum states. We do not possess complete information about the states to be discriminated. Our devices have two inputs, one for the qubit whose identity is to be determined, and the other for the copies of one or both of the possible states that it can be in. In the case that only one of the states is provided, it is assumed that the other state is known, and this knowledge is built into the device. The states sent into the second input can be regarded as a program. To change the set of states between which we are discriminating, we do not have to change the device, but merely supply it with a different program.

We want to point out a striking feature of the programmable state discriminators in which copies of both of the states to be discriminated are provided. Neither the optimal detection operators nor the boundaries for their region of validity depend on the unknown states. Therefore, these devices are *universal*, they will perform optimally for any set of unknown states. Only the probability of success for fixed but unknown states will depend on the overlap of the states. However, both this expression and its average over all possible inputs is optimal.

The devices described here demonstrate the role played by *a priori* information. All of them have a smaller success

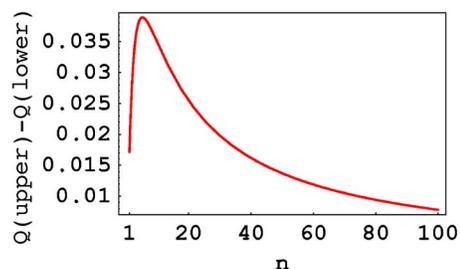


FIG. 4. (Color online) The difference between the upper curve and the lower curve in Fig. 3 as a function of n . The difference between the performance of the PVM and POVM is maximum for $n=5$.

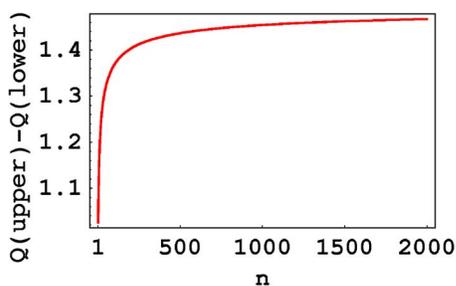


FIG. 5. (Color online) The ratio of the upper and lower curve in Fig. 3 as a function of n . Asymptotically the failure rate of the PVM is 50% higher than that of the POVM.

probability than the one designed for a case in which we know both of the input states, and the device for two unknown input states has a smaller success probability than the one designed for the case when we know one of the input states. There is a trade off between flexibility and success probability. The greater the amount of information about the states that is carried by a quantum program, the smaller the probability of successfully discriminating between the states, but the larger the set of states for which the device is useful. This flexibility suggests that programmable discriminators will be useful as parts of larger devices that produce quantum states that need to be identified.

We conclude our paper by summarizing what we know about programmable discriminators with quantum programs in which the programs consist of copies of the states to be discriminated. The most general problem of this type is when we have n_A copies of the state of the program system A , n_C copies of the state of the program system C , and n_B copies of the state of the data system B . In this case, the task is to discriminate two input states

$$|\Psi_1^{\text{in}}\rangle = |\psi_1\rangle_A^{\otimes n_A} |\psi_1\rangle_B^{\otimes n_B} |\psi_2\rangle_C^{\otimes n_C},$$

$$|\Psi_2^{\text{in}}\rangle = |\psi_1\rangle_A^{\otimes n_A} |\psi_2\rangle_B^{\otimes n_B} |\psi_2\rangle_C^{\otimes n_C}, \quad (7.1)$$

where the subscripts A and C refer to the program registers (A contains $|\psi_1\rangle$ and C contains $|\psi_2\rangle$), and the subscript B refers to the data register. Our goal would be to optimally distinguish between these inputs, keeping in mind that one has no knowledge of $|\psi_1\rangle$ and $|\psi_2\rangle$ beyond their *a priori* probabilities. The problem in which the number of copies of the program states are equal and greater than one, but we have only one copy of the data state solved for equal *a priori* probabilities [19]. The problem in which we have only one copy of each program state, but an arbitrary number of copies of the data state has been solved here. The general problem remains open.

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