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A simple test of quantumness for a single system

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Abstract
We propose a simple test of quantumness which can decide whether for a given set of accessible experimental data the classical model is insufficient. Take two observables A, B such that for all states ψ their mean values satisfy $0 \leqslant \langle\psi|A|\psi\rangle \leqslant \langle\psi|B|\psi\rangle \leqslant 1$. If there exists a state ϕ such that the second moments fulfill the inequality $\langle\phi|A^2|\phi\rangle > \langle\phi|B^2|\phi\rangle$ then the system cannot be described by a classical probabilistic scheme. An example of an optimal triple (A, B, ϕ) in the case of a qubit is given.

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Although we are confident that the proper theory describing all physical phenomena is the quantum theory, there are many situations where the classical description in terms of functions and probability distributions over a suitable 'phase–space' is sufficient. In particular, the systems consisting of a large number of particles and/or emerging in quantum states characterized by large quantum numbers are supposed to behave classically. The standard explanation of this fact refers to the stability properties of quantum states with respect to the interaction with an environment. For large quantum systems the interaction with the environment is so strong that most quantum states rapidly decay (decohere) and the remaining manifold of experimentally accessible states can be described by classical models. However, the actual border between quantum and classical worlds is still a topic of theoretical debate and experimental efforts [1].

For example, consider a Josephson junction in the regime of parameters corresponding to the so-called *superconducting qubit*. Increasing evidence shows that the experimental data agree with the theoretical model of 2-level quantum system [2]. On the other hand, there are strong arguments that at least most of the experimental data are consistent with the classical model of perturbed damped nonlinear oscillator [3]. Another example of systems which are believed to be useful for quantum information processing but on the other hand might live in a semiclassical regime are Rydberg atoms [4]. Therefore, it is important to find a simple operational (i.e. to a large extent model independent) test which could exclude the classical model and could be applied to a single system. It is generally believed that the simplest operational and model independent test of quantumness (or strictly speaking non-classicality) is based on Bell inequalities [5]. This test is, however, more difficult to realize because it involves a system composed of two parts which should be well separated to avoid the *locality loophole*.

More precisely, our test is designed to exclude the *classical algebraic model* which is a special case of the general *algebraic model* (AM). AM is an abstract formalism in which (bounded) observables are elements of a certain C^*-algebra $\mathcal{A}$, and states are positive normalized functionals $\mathcal{A} \ni A \mapsto \langle A\rangle_\rho$ with $\langle A\rangle_\rho$ denoting the mean value of an observable A in a state ρ. The algebraic structure is encoded into the following assumptions:

(a) for any two observables identified with elements $A, B \in \mathcal{A}$ there exists an observable which can be identified with $A + B$,

(b) the kth moment of the observable A is given by $\langle A^k\rangle$ where A^k is a k-power of A in the algebra $\mathcal{A}$.

For all practical purposes we can restrict ourselves to two extreme cases: the first being a classical AM where $\mathcal{A}$ is an algebra of functions on a certain 'phase–space' with $\langle A\rangle_\rho = \int A(x)\rho(x)\,\mathrm{d}x$, where $\rho(x)$ is some probability distribution, and the second being a finite quantum AM in which $\mathcal{A}$ is an algebra of matrices and $\langle A\rangle_\rho = \mathrm{Tr}(\rho A)$, where ρ is some density matrix. For any pair of observables $A, B \in \mathcal{A}$, the order relation $A \leqslant B$ means that $\langle A\rangle_\rho \leqslant \langle B\rangle_\rho$ for all states ρ (in fact, it is enough to take all pure states).

One should stress that the proposed simple test does not exclude the general *hidden variable models* (HV) which do not possess an algebraic structure superimposed on the set of observables (see an exhaustive discussion of HV models in [6]). The examples of such models, for which the similar questions of 'classicality' have been discussed, were those based on *unsharp measurements* realized by POVMs [7], experimental events described by *effects* [8] or *quantum logic* [9]. As our aim is not a discussion of an ultimate model of Nature but rather a practical test which could eliminate a well-defined classical theory for some specific systems the restriction to AM seems reasonable.

The following result can be proved within the general formalism of the AM [10].

Theorem. *The following implication*

$$0 \leqslant A \leqslant B \Longrightarrow A^2 \leqslant B^2 \tag{1}$$

always holds if and only if the algebra $\mathcal{A}$ is commutative, i.e. isomorphic to the algebra of continuous functions on a certain compact space.

As a consequence of the above theorem, for any quantum system there exists a pair of observables (identified with matrices) (A, B) such that the eigenvalues of A, B and $B - A$ are nonnegative but the matrix $B^2 - A^2$ possesses at least one negative eigenvalue. One should add that instead of the square function $A \rightarrow A^2$ one can take any *non-operator monotone function* what might be useful for the applications.

To apply this mathematical result we assume that the experimental situation can be described in terms of the set $\mathcal{S}_{\rm exp}$ of accessible initial states of a certain physical system and the set of accessible measurements (observables) $\mathcal{A}_{\rm exp}$. For any observable $A \in \mathcal{A}_{\rm exp}$ and any state $\rho \in \mathcal{S}_{\rm exp}$ we can extract (by repeating measurements on the fixed initial state ρ) the statistics of the measurement outcomes. Therefore, if $A \in \mathcal{A}_{\rm exp}$ then for any continuous real function F, $F(A) \in \mathcal{A}_{\rm exp}$, and can be measured by the same apparatus as A.

We say that the pair $(\mathcal{A}, \mathcal{S})$, where $\mathcal{A}$ is a C^*-algebra and $\mathcal{S}$ is a set of linear, positive and normalized functionals on $\mathcal{A}$, is a *minimal algebraic model* for our set of experimental data if:

(1) we can identify $\mathcal{A}_{\rm exp}$ with a subset of $\mathcal{A}$ and $\mathcal{S}_{\rm exp}$ with a subset of $\mathcal{S}$, such that the corresponding mean values reproduce experimental data,

(2) for any pair of observables $A, B \in \mathcal{A}_{\rm exp}$, $\langle A\rangle_\psi \leqslant \langle B\rangle_\psi$ for arbitrary $\psi \in \mathcal{S}_{\rm exp}$ implies $\langle A\rangle_\phi \leqslant \langle B\rangle_\phi$ for arbitrary $\phi \in \mathcal{S}$.

According to the definition above, if we can find two accessible observables A and B such that for all accessible states $0 \leqslant \langle A\rangle \leqslant \langle B\rangle$ and if we can prepare a certain state ρ satisfying $\langle A^2\rangle_\rho > \langle B^2\rangle_\rho$, then we can say that the set of experimental data does not admit a minimal classical model.

One could still argue that there may exist a non-minimal classical AM describing the data ('minimality loophole'). In this case, the classical observable $B - A$ possesses negative outcomes (values of the function) which are not detectable by the differences of averages $\langle B\rangle - \langle A\rangle$. This means that the accessible states (probability distributions) are too coarse grained in comparison to the accessible observables. This is not a reasonable assumption as we typically use the same technology for state preparation and for observable measurements. Hence, we expect a similar resolution for both types of processes.

In practical situations, we would like to test a definite quantum model for which we have some arguments for how to associate observables and states with definite experimental procedures. Therefore, instead of a random guess it is useful to find the examples of triples (A, B, ϕ) which maximally violate classicality and can be used to optimally design the experimental setting. This can easily be achieved in the case of a qubit which is the most important example for quantum information.

We search for a pair of 2×2 matrices A, B and a pure state ϕ which satisfy

$$0 \leqslant A \leqslant B \leqslant I, \qquad \langle A^2\rangle_\phi \equiv \langle\phi|A^2|\phi\rangle > \langle B^2\rangle_\phi \equiv \langle\phi|B^2|\phi\rangle, \tag{2}$$

where the observables A and B are normalized in such a way that their upper bound is the identity. Such a triple is given as an example as

$$A = \begin{pmatrix} a_1 & \xi \\ \xi^* & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \qquad \phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3}$$

The matrix B is chosen to be diagonal, with the identity as its upper bound, and this choice of basis can be made since the solution to this problem is unique up to unitary equivalence. From the upper bound it can be seen that both eigenvalues of B should be at most 1, and to maximize the violation of (1) one of the eigenvalues is chosen to be fixed at 1. The usual condition $|\alpha|^2 + |\beta|^2 = 1$ applies for the parameters of the state ϕ. The positivity of the observables A, B and $(B - A)$ is ensured by the requirement that both their diagonal elements and determinants are positive. These conditions are expressed below as

$$0 \leqslant b \leqslant 1, \qquad 0 \leqslant a_1 \leqslant 1, \tag{4}$$

and

$$0 \leqslant a_1 a_2 - |\xi|^2, \qquad 0 \leqslant (1 - a_1)(b - a_2) - |\xi|^2. \tag{5}$$

The lower eigenvalue of the matrix $(B^2 - A^2)$ is found to be

$$\tfrac{1}{2}\Big(b^2 + 1 - a_1^2 - a_2^2 - 2|\xi|^2 - \sqrt{\big(b^2 - 1 + a_1^2 - a_2^2\big)^2 + 4(a_1 + a_2)^2|\xi|^2}\Big). \tag{6}$$

A numerical technique is used to calculate the maximal violation of the inequality (1) since we are unable to find an exact solution to this optimization problem. In finding the optimal set of parameters a_1, a_2, b and real ξ, it can be seen that the maximal violation of the inequality (1) arises when conditions (5) are equalities rather than inequalities. This reduces the problem from four unknowns to two, and the triplet can then be written as

$$A = \begin{pmatrix} a_1 & \sqrt{a_1 a_2} \\ \sqrt{a_1 a_2} & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{a_2}{1-a_1} \end{pmatrix}, \qquad \phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{7}$$

The parameters which result in one of the eigenvalues of $(B^2 - A^2)$ attaining it's most negative value, and thus maximally violating inequality (1), while still remaining within the constraints give us the triplet

$$A = \begin{pmatrix} 0.724 & 0.249 \\ 0.249 & 0.0854 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0.309 \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0.391 \\ 0.920 \end{pmatrix}. \tag{8}$$

In this example, the value of $\langle\phi|B|\phi\rangle - \langle\phi|A|\phi\rangle$ is 0.0528, a positive value, whereas it can be seen that $\langle\phi|B^2|\phi\rangle - \langle\phi|A^2|\phi\rangle = -0.0590$ which clearly demonstrates the quantum nature of this example. The eigenvectors and corresponding eigenvalues of A using these parameters is calculated to be

$$A \begin{pmatrix} 0.946 \\ 0.325 \end{pmatrix} = 0.809 \begin{pmatrix} 0.946 \\ 0.325 \end{pmatrix} \tag{9}$$

and

$$A \begin{pmatrix} -0.325 \\ 0.946 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

To give a concrete example, one can apply these results to the polarization of a single photon which could be the first feasibility test of our method. Choosing a polarization basis as $|H\rangle, |V\rangle$ and attributing the values 1 to $|H\rangle$ and 0.309 to $|V\rangle$ we obtain the observable B. The observable A corresponds to a rotated polarization basis $|H'\rangle = \cos(19^\circ)|H\rangle + \sin(19^\circ)|V\rangle$, $|V'\rangle = -\sin(19^\circ)|H\rangle + \cos(19^\circ)|V\rangle$ with the eigenvalues 0.809 and 0, respectively. The maximal violation of classicality should be observed in the neighborhood of the state $|\phi\rangle = \cos(67^\circ)|H\rangle + \sin(67^\circ)|V\rangle$. Obviously, more interesting experiments could be done for systems with a still questionable quantum character [11].

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