CLUSTER STATES & QUANTUM COMPUTING

Note Title 9/12/2008

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1/ Review of Standard Circuit Model

2/ Introduce Cluster Quantum Computing

3/ Advanced Topics

1/ Review of Circuit Model

→ Start with 2-state quantum systems

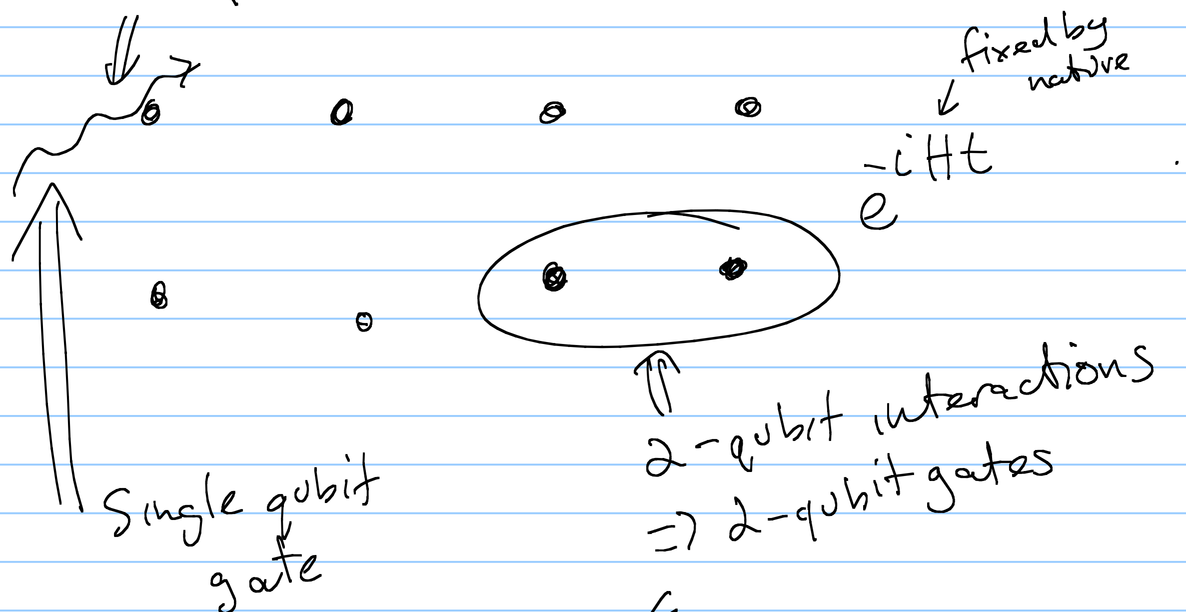
$|0\rangle \quad |1\rangle$

$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}$

e.g. Spin-½ particle $|\sigma, m\rangle$

$|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

classical field $|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$



$$U = e^{-iHt}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

I / X

CNOT (controlled-NOT)

1/ ~~B~~ Start with product

2/ Algorithm: Choose single qubit gates & 2-qubit gates

$\Rightarrow$ Evolve to some entangled state

n-qubits $\quad |\Psi\rangle \sim 2^n$ complex entries

3/ Measurement (wavefunction collapse)

Notation

$X, Y, Z \Rightarrow$ Pauli matrices

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\leftarrow$ diagonal

$$CNOT = \underbrace{\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}}_{\text{computational basis}}$$

$\Leftarrow$ if ~~qubit 1~~ qubit 1 is in the state $|1\rangle$ then qubit 2 gets flipped $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$

$|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$

$$C\text{-}Z = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

π-phase

$\Uparrow$

$|1\rangle|1\rangle$

Exercise Show that if we write the C-Z gate in this basis:

$|0\rangle|+\rangle, |0\rangle|-\rangle, |1\rangle|+\rangle, |1\rangle|-\rangle$

(where $|\pm\rangle = \dfrac{|0\rangle \pm |1\rangle}{\sqrt{2}}$)

then

$$C\text{-}Z = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$$

(ie looks like a CNOT gate)

$|0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle$

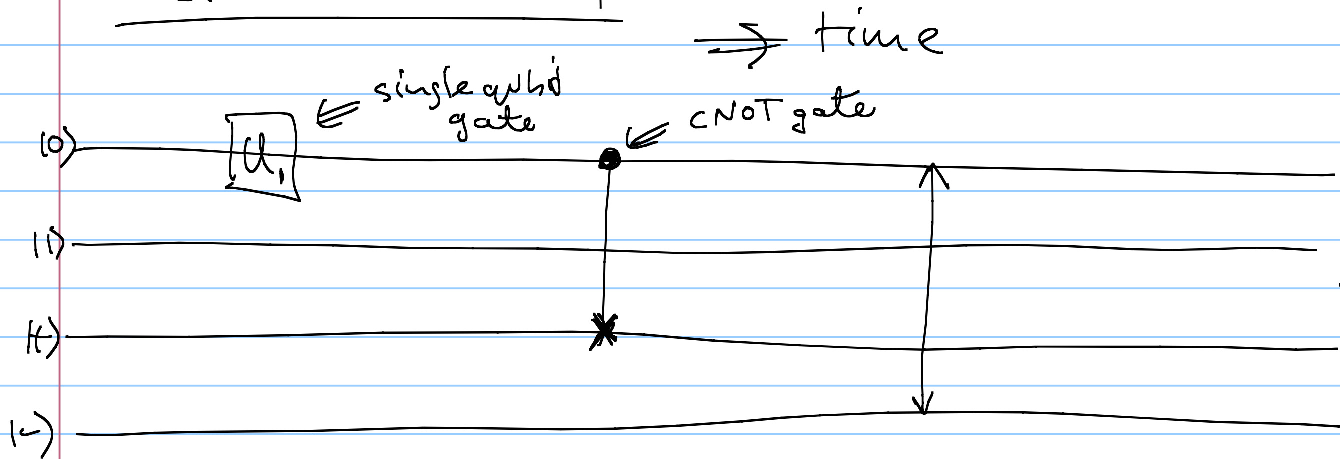
$$C\text{-}Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$|0\rangle|+\rangle \to |0\rangle|+\rangle$
$|0\rangle|-\rangle \to |0\rangle|-\rangle$
$|1\rangle|+\rangle \to |1\rangle|-\rangle$
$|1\rangle|-\rangle \to |1\rangle|+\rangle$

$|-\rangle =$

(Symmetric $\Rightarrow$ could write qubit 1 in the $|\pm\rangle$ basis & 2 in the $|0\rangle, |1\rangle$ basis & get the same thing

Circuit Notation



Universality single qubit gates & two qubit gates can be used to build / evolve any unitary matrix

(Key question: what interesting unitaries can be built using poly(n) such gates?)
n = # of qubits.

Big Picture

Product State $\Rightarrow$ ~ entangled
$|0\rangle\otimes|0\rangle\otimes\cdots|0\rangle$
$\Downarrow$
Single qubit Measurements

$|0\rangle, |1\rangle \approx$ eigenstates of Z

$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}$

$|+\rangle, |-\rangle$ " " X

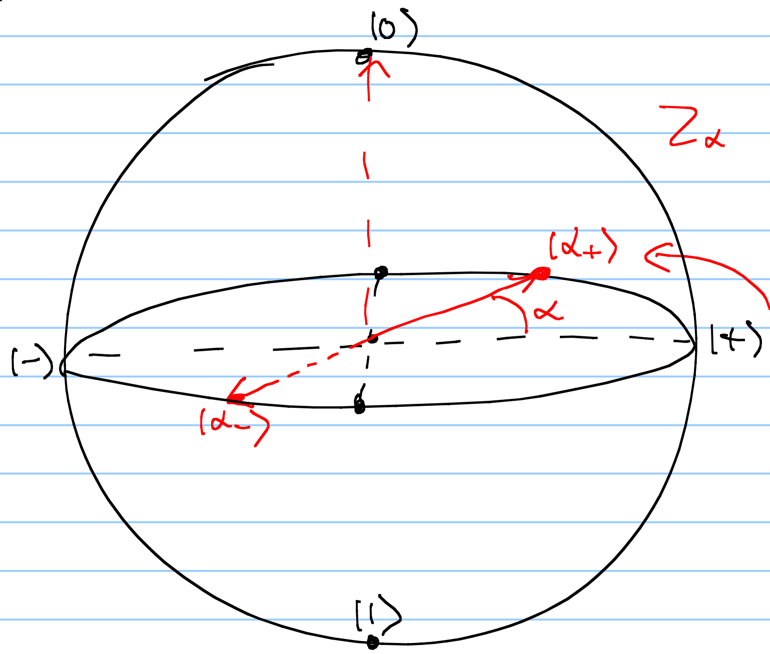
("X-measurement")

$|\alpha_+\rangle = Z_{-\alpha}|+\rangle$

$$|\alpha_\pm\rangle \equiv e^{i\alpha/2}|0\rangle \pm e^{-i\alpha/2}|1\rangle$$

$(|0\rangle + |1\rangle)$

$Z_\alpha \quad e^{-i\alpha/2}|0\rangle + e^{i\alpha/2}|1\rangle$



golay code

Other useful gates

$$H = \frac{1}{\sqrt{2}} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{array}{l} |0\rangle \to |+\rangle \\ |1\rangle \to |-\rangle \end{array}$$

$$Z_\alpha \equiv e^{-i\alpha/2 \cdot Z} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}$$

$$X_\beta \equiv e^{-i\beta/2 \cdot X} = \underbrace{\begin{pmatrix} \cos\beta/2 & -i\sin\beta/2 \\ -i\sin\beta/2 & \cos\beta/2 \end{pmatrix}} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$n =$ no. of qubits

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \cdots$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2^n dimensional vector

$$\begin{pmatrix} U \\ 2^n \times 2^n \text{ matrix} \\ \\ \\ \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

Section 2 — Cluster State Computation

(MBQC - measurement based quantum computing)

Big Picture

Start with an Entangled State (fixed) $\Rightarrow$ Apply single qubit measurements (determine the algorithm)

C.M : $|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$

C.M 1 & 2-qubit gates determine the algorithm

$\Downarrow$

Final set of computational Basis measurements $\Rightarrow$ Outcome

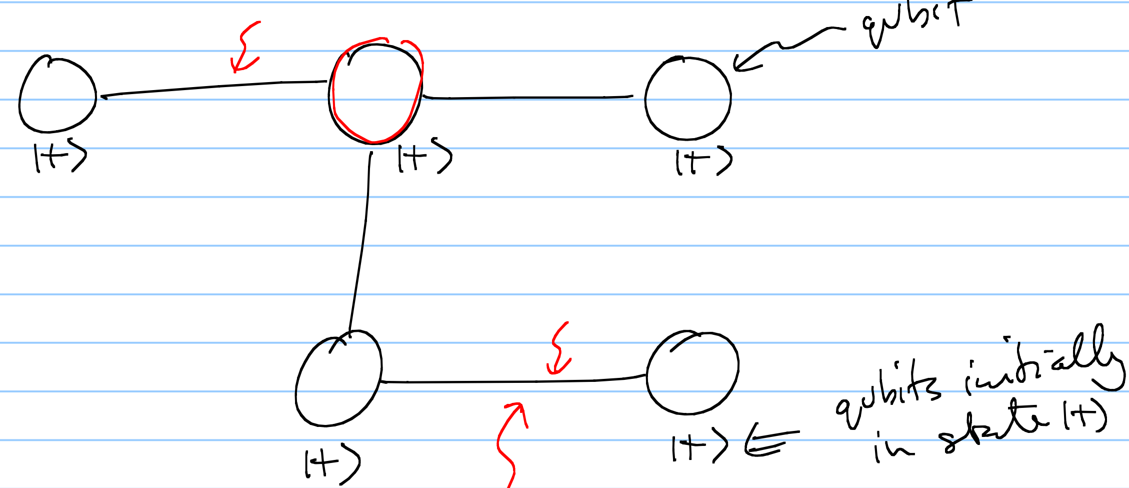
If the circuit model used n qubits & L "gates" (time steps)

for cluster state computing we require an initial entangled state of $\sim (n \times L)$ qubits

L: poly(n)

What is this magical state

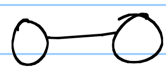
Graphical Representation



1/ Begin in $|+\rangle$

2/ Apply CZ gates along bonds

apply the C-Z $= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ between connected qubits

e.g.  (2-qubit linear cluster)

$(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$ (drop $1/\sqrt{2}$)

$= |00\rangle + |01\rangle + |10\rangle + |11\rangle$

$\xRightarrow{C-Z} |00\rangle + |01\rangle + |10\rangle - |11\rangle$ (entangled state)

$\equiv \underbrace{|0+\rangle + |1-\rangle = |+0\rangle + |-1\rangle}_{\text{Bell states}}$

$|a\, b\rangle + |\bar{a}\, \bar{b}\rangle$

$|00\rangle + |11\rangle = |\phi^+\rangle$

Exercise: Show equivalent to starting with $|\phi^+\rangle$ & applying a Hadamard.

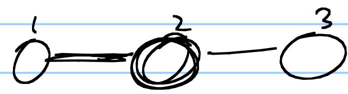
Recall exercise 1

$$|+\rangle|+\rangle$$

$$CZ: \ (|0\rangle+|1\rangle)|+\rangle$$

$$= |0\rangle|+\rangle + |1\rangle|-\rangle$$

e.g 2



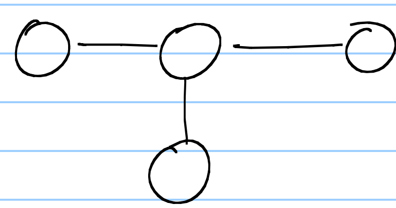
initial state $\quad |+\rangle \quad |+\rangle \quad |+\rangle$

$$|+\rangle\left(|0\rangle+|1\rangle\right)|+\rangle$$

Apply CZ_{12}: $\left(|+\rangle|0\rangle+|-\rangle|1\rangle\right)|+\rangle$

CZ_{23} $\left(|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle\right)$ $\quad \equiv |L_3\rangle$ (linear 3)

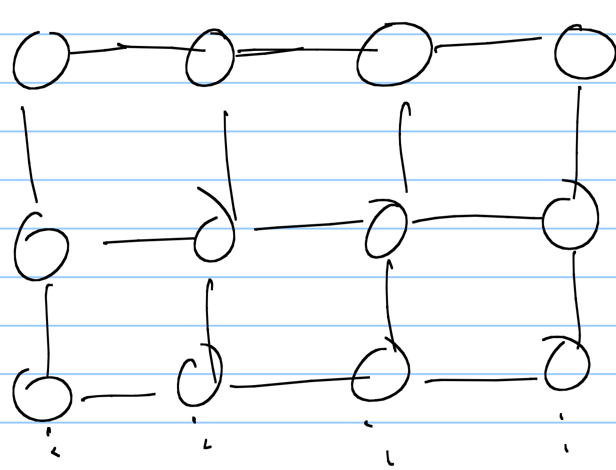
Exercise 3 Show the state



is $\quad |+0++\rangle + |-1--\rangle$

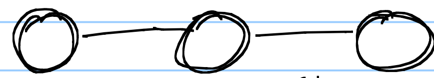
Universal Cluster State

(Raussendorf & Briegel)

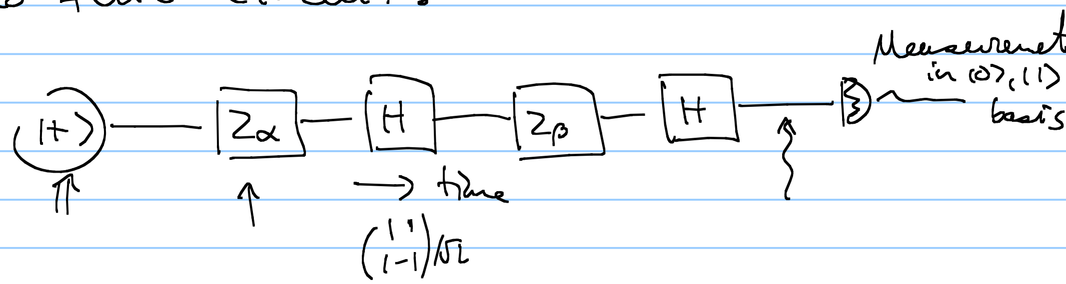


How to use a cluster state?

We're going to show how to use



to do this circuit:



First compute what the output of the circuit is:

$$|\psi\rangle = \underbrace{H\, Z_\beta\, H}_{H Z_\beta H \equiv X_\beta}\, Z_\alpha\, |+\rangle$$

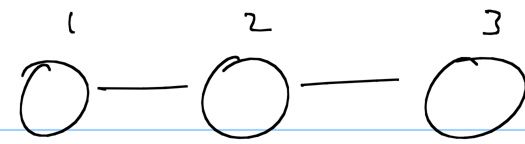
$$|\psi\rangle = X_\beta Z_\alpha |+\rangle$$

$$= \begin{pmatrix} \cos\beta/2 & -i\sin\beta/2 \\ -i\sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} /\sqrt{2}$$

$$= \begin{pmatrix} \cos\beta/2\, e^{-i\alpha/2} & i\sin\beta/2\, e^{i\alpha/2} \\ -i\sin\beta/2\, e^{-i\alpha/2} & + \cos\beta/2\, e^{i\alpha/2} \end{pmatrix} /\sqrt{2}$$

$$\equiv \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\text{Prob}(|0\rangle) = |A|^2$$



$$|+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle$$

(i) Measure qubit 1 in

$$\{\, |\alpha_+\rangle\langle\alpha_+| ,\ |\alpha_-\rangle\langle\alpha_-| \,\}$$

$$|\alpha_\pm\rangle = e^{i\alpha/2}|0\rangle \pm e^{-i\alpha/2}|1\rangle$$

<u>Exercise</u> Show $|\alpha_\pm\rangle = \cos\alpha/2\,|\pm\rangle + i\sin\alpha/2\,|\mp\rangle$

$$\langle\alpha_\pm| = \cos\alpha/2\,\langle\pm| - i\sin\alpha/2\,|\mp\rangle$$

<u>Imagine we get the α_+ outcome:</u> (what is the probability of this)

$$\underset{\left(\cos\alpha/2\,\langle+| - i\sin\alpha/2\,\langle-|\right)}{\langle\alpha_+|}\Big(|+0+\rangle + |-1-\rangle\Big)$$

$$|\alpha_+\rangle\big(\langle\alpha_+|\,|\psi\rangle\big) \qquad \cos\alpha/2\,|0+\rangle - i\sin\alpha/2\,|1-\rangle$$

(ii) Measure qubit 2 in $\{\, |\beta_+\rangle\langle\beta_+| ,\ |\beta_-\rangle\langle\beta_-| \,\}$

Imagine I get the $|\beta_+\rangle\langle\beta_+|$ outcome:

$$\underset{\left(e^{-i\beta/2}\langle 0| + e^{i\beta/2}\langle 1|\right)}{\langle\beta_+|}\ \cos\alpha/2\,|0+\rangle - i\sin\alpha/2\,|1-\rangle$$

$$\underbrace{e^{-i\beta/2}\cos\alpha/2\,|+\rangle - i\sin\alpha/2\cdot e^{i\beta/2}\,|-\rangle}_{\text{state of qubit 3 after the collapses.}}$$

<u>Claim</u> equivalent to $\begin{pmatrix} A \\ B \end{pmatrix}$ from the circuit.

$$= \begin{pmatrix} \cos\beta/2\, e^{-i\alpha/2} & i\sin\beta/2\, e^{i\alpha/2} \\ -i\sin\beta/2\, e^{-i\alpha/2} & +\cos\beta/2\, e^{i\alpha/2} \end{pmatrix} / \sqrt{2}$$

$$\equiv \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\text{Prob}(|0\rangle) = |A|^2$$

$$e^{-i\beta/2} \cos\alpha/2\, |+\rangle - i\sin\alpha/2 \cdot e^{i\beta/2}\, |-\rangle$$

Exercise: Show these states are equivalent.

~~S~~ Story so far

If you are lucky enough to get first the $|\alpha+\rangle$ outcome & then $|\beta+\rangle$ outcome you will have the same output as from the circuit computation

$\Rightarrow$ Prob of this is $1/4$

for a longer chain would decrease exponentially of always getting lucky.

Exercise: Show that on any cluster state the probability of any ~~$\mathbb{Z}$~~ measurement on the initial qubit is $1/2$.

Final Exercise

Compute the output state if

on the first measurement on $|L_3\rangle$ I got $|\alpha_+\rangle\langle\alpha_+|$

but on the second qubit I got $|\beta_-\rangle\langle\beta_-|$

How is this state related to $|\psi\rangle = \begin{pmatrix} A \\ B \end{pmatrix}$

the desired output state.

(what unitary would transform the output state to $|\psi\rangle$?)