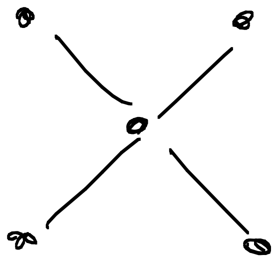
Notiztitel 27.09.2008

Graph states

+ Gottesman-Knill



I) How to prepare graph states

II) Graph states and stabilizer states

III) Gottesman-Knill Theorem.

I) $\underline{n=2}$

1) qubit

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

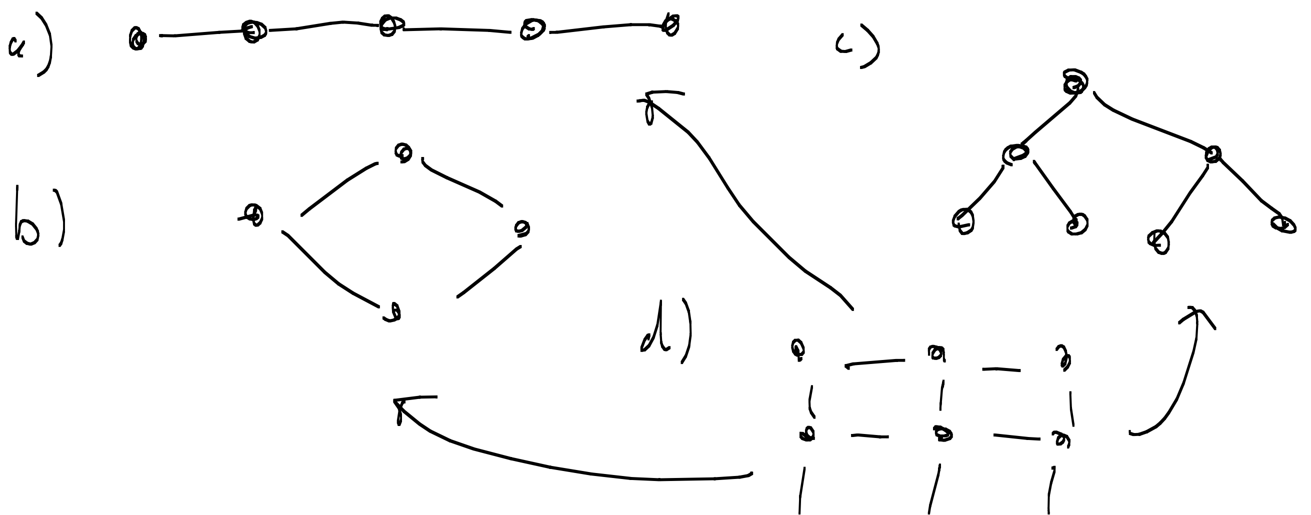
2)

controlled-Z gate

$$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$$
$$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$$
$$|1\rangle|0\rangle \longmapsto |1\rangle|0\rangle$$
$$|1\rangle|1\rangle \longmapsto -|1\rangle|1\rangle$$

$$|+\rangle \otimes |+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$



II) Graph states are stabilizers

1)  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$g_1 = X \qquad S = \mathbb{1}, X$

$|+\rangle = |+\rangle \otimes |+\rangle$ $\qquad g_1 = X \otimes \mathbb{1}$, $\quad g_2 = \mathbb{1} \otimes X \qquad S = \langle g_1, g_2 \rangle$

$CZ|+\rangle \qquad CZ \cdot CZ = \mathbb{1}$

$g \in S \Rightarrow g|+\rangle = |+\rangle$

$\Rightarrow g\, CZ \cdot CZ |+\rangle = |+\rangle$

$\Rightarrow (CZ\, g\, CZ)\, CZ|+\rangle = CZ|+\rangle$

$\Rightarrow CZ|+\rangle$ is eigenvector of

$CZ\, g\, CZ.$

$$C\text{-}Z \; \mathbb{1} \otimes X \; C\text{-}Z$$

$$= C\text{-}Z \left(|0\rangle\langle 0| \otimes X \right) C\text{-}Z$$

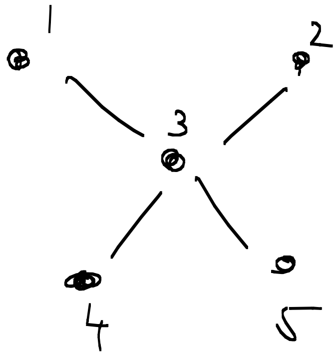
$$+ \, C\text{-}Z \left(|1\rangle\langle 1| \otimes X \right) C\text{-}Z$$

$$= |0\rangle\langle 0| \otimes X - |1\rangle\langle 1| \otimes X$$

$$= Z \otimes X .$$

$$g_1 = \mathbb{1} \otimes X \qquad \leadsto \qquad g_1' = Z \otimes X$$

$$g_2 = X \otimes \mathbb{1} \qquad\qquad\qquad g_2' = X \otimes Z$$



$g_1 = X \; \mathbb{1} \; \mathbb{1} \; \mathbb{1} \; \mathbb{1}$

$g_2 = \mathbb{1} \; X \; \mathbb{1} \; \mathbb{1} \; \mathbb{1}$

$\vdots$

$g_5 = \mathbb{1} \; \mathbb{1} \; \mathbb{1} \; \mathbb{1} \; X$

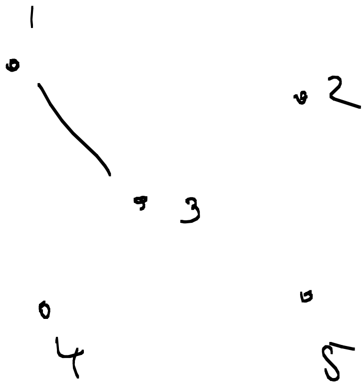
$g_1 = X \; \mathbb{1} \; Z \; \mathbb{1} \; \mathbb{1}$

$g_2 = \mathbb{1} \; X \; \mathbb{1} \; \mathbb{1} \; \mathbb{1}$

$g_3 = Z \; \mathbb{1} \; X \; \mathbb{1} \; \mathbb{1}$

$g_4 = \dots$

$g_5 = \dots$



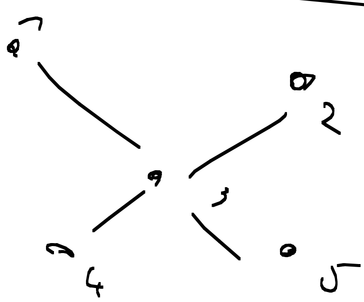
$g_1 = X \; \mathbb{1} \; Z \; \mathbb{1} \; \mathbb{1}$

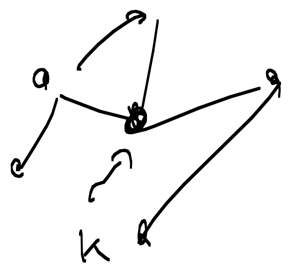
$g_2 = \mathbb{1} \; X \; Z \; \mathbb{1} \; \mathbb{1}$

$g_3 = Z \; Z \; X \; Z \; Z$

$g_4 = \mathbb{1} \; \mathbb{1} \; Z \; X \; \mathbb{1}$

$g_5 = \mathbb{1} \; \mathbb{1} \; Z \; \mathbb{1} \; X$





$$g_k = X_k \cdot \bigotimes_{\underbrace{i \in Neig(k)}_{\text{neighbors of } k}} Z_i$$

$g_k \, g_l$

III) Gottesman – Knill Theorem

$$(C\text{-}Z)^\dagger = C\text{-}Z.$$

$$U X U^\dagger = Z$$

Def.: U is an element of the Clifford group if it maps Pauli operators to Pauli operators under ~~sandwiching.~~ conjugation.

$\leadsto$ • $|\Psi\rangle$ is a stabilizer state w/ stabilizer group $S = \langle g_1, \dots, g_m \rangle$

• U is Clifford

$\Rightarrow U|\Psi\rangle$ with stabilizer group

$$S' = \langle U g_1 U^\dagger, \ldots, U g_m U^\dagger \rangle .$$

$\rightsquigarrow$ The Gottesmann-Knill-Corollary.

<u>Ex.:</u> 1) C-Z $\in$ Clifford

2) <u>Hadamard</u> $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in$ Clifford

$\hookrightarrow H X H^\dagger = Z$

$H Z H^\dagger = X$

3) Phase gate $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$\exp\{i \frac{\pi}{4}\}$ 8

<u>Result:</u> Clifford group is <u>generated</u> by $\{C\text{-}Z, H, S\}$.

<u>R. Werner</u> If $[U_{1,2}, U_{2,3}] = 0$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{i\varphi} \end{pmatrix}$$

$\Rightarrow$ up to conjugation they are of this form.

1) 

Similar to $(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}$

$g_1 = X \otimes Z$

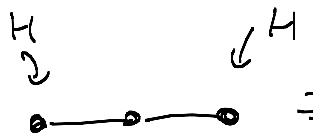
$g_2 = Z \otimes X$

H: exchanges $Z \leftrightarrow X$

$\rightsquigarrow g_1 = X \otimes X$

$g_2 = Z \otimes Z$

 $\equiv_{LU} (|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}$

2) 

$\equiv_{LU} |GHZ\rangle = (|000\rangle + |111\rangle)\frac{1}{\sqrt{2}}$

$g_1 = X Z \mathbb{1} \rightsquigarrow g_1' = Z Z \mathbb{1}$

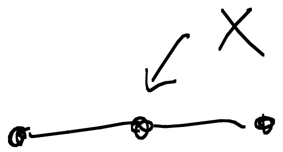
$g_2 = Z X Z \rightsquigarrow g_2' = X X X$

$g_3 = \mathbb{1} Z X \rightsquigarrow g_3' = \mathbb{1} Z Z$

$\square$

$\to \left(|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle \right) / \sqrt{8}$

IV Local meas. on graphs

 $= |\Psi\rangle$

$$\begin{array}{ccc} |+\rangle & & |-\rangle \\ \downarrow & & \downarrow \\ \langle +|_2 |\Psi\rangle & & \langle -|_2 |\Psi\rangle \\ \downarrow & & \\ \langle +|_2 \; |\Psi\rangle\langle\Psi| \; |+\rangle & & \end{array}$$

$$\begin{aligned} = \langle +|_2 \; & \underline{\mathbb{1}\,\mathbb{1}\,\mathbb{1}} \\ & + \underline{X\, Z\, \mathbb{1}} \\ & + \underline{Z\, X\, Z} \\ & + \underline{\mathbb{1}\, Z\, X} \\ & + \underline{iZ\, Y\, Y} \\ & - \underline{Y\, Y\, \mathbb{1}} \\ & + \underline{X\, \mathbb{1}\, X} \\ & - \underline{Y\, X\, Y} \end{aligned} \quad |+\rangle_2$$

$\langle Y_+|Z|Y_+\rangle = 0$

$\langle +|Z|+\rangle = 0$

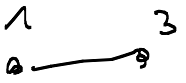
$\langle +|X|+\rangle = 1$

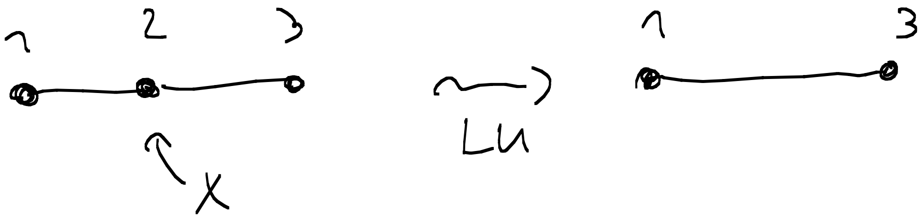
$Y := |Y_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$$= \mathbb{1}_1 \otimes \mathbb{1}_3 + Z \otimes Z + X \otimes X - Y \otimes Y$$

$$= \left(|00\rangle + |11\rangle \right) \left(\langle 00| + \langle 11| \right) \tfrac{1}{2}$$

$\underset{LU}{\cong}$

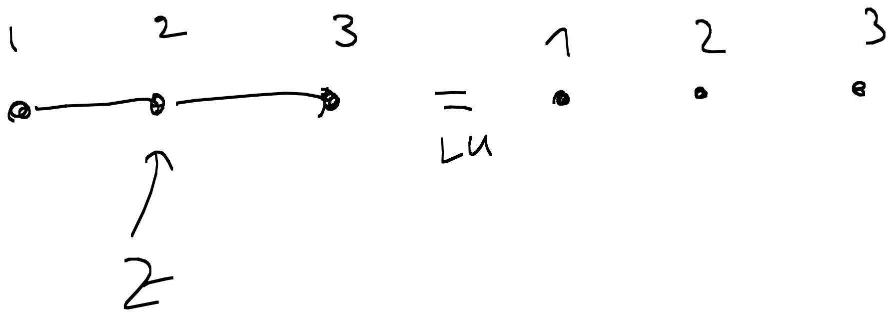


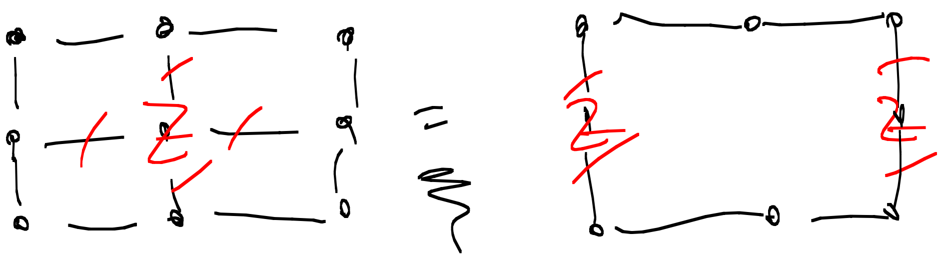


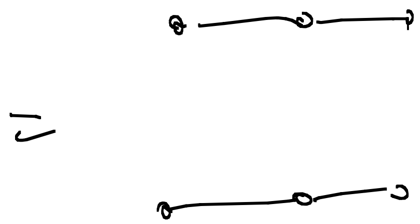
$$\boxed{\text{entanglement localization}}$$

Localizable entanglement between sites 1, 3 := max. ent. I can create between 1 & 3 by measuring in the middle.

Z measurements

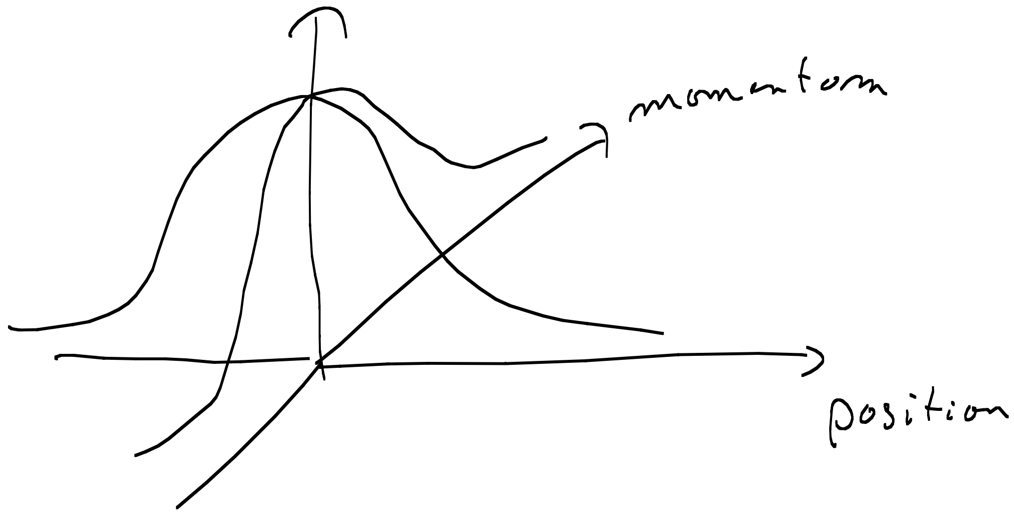






phase space

classical



<u>QM</u> described by $\Psi(x)$

quasi-phase-space descripti. of q. particles

→ Wigner functions.

two symmetries: shift
& acceleration.

$$S_y \Psi(x) \mapsto \Psi(x-y)$$

qubit shift operator

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$\mathcal{F}(X)\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

C.V. gaussian state $\mathbb{R}$

$$\Psi(x) \propto \exp\{-i x \theta x + y x\}$$

$\mathbb{R}$

qu-dit stab. state

$$\langle x|\Psi\rangle = \Psi(x) \propto \exp\{-i x \theta x + y x\}$$

$\in \mathbb{Z}_d$